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Prof. F. J. M. Stratton, President, in the Chair.

The election by the Council of the following Fellow was duly confirmed :—

William Rhys Evans, B.Sc., A.M.I.E.E., Parkfield, Cannock Road, Stafford.

Seventy-nine presents were announced as having been received since the last meeting, including :—

Greenwich, Royal Observatory, Astrographic Catalogue, vol. vi ;
Sir James Jeans, "The New Background of Science" ;
H. Macpherson, "Makers of Astronomy" ;
H.M. Nautical Almanac Office, "Planetary Co-ordinates for the years 1800–1940, referred to the equinox of 1950·0" (presented by Dr. Comrie).

THE EQUILIBRIUM OF DISTORTED POLYTROPES.

(IV) The Rotational and the Tidal Distortions as Functions of the Density Distribution.

S. Chandrasekhar.

§ 1. The formal mathematical theory of the equilibrium of (i) the rotationally distorted polytropes, (ii) the tidally distorted polytropes, and (iii) the rotationally and tidally distorted polytropes, which are components of double-star systems, have already been worked out in three recent

communications.* In this paper we shall more closely examine the general properties of these three classes of distorted polytropes and see what information can be obtained regarding the general nature of the functional dependence of the rotational and tidal distortions on the complete march of the density distribution in the undistorted configuration.

In this connection it is of interest to consider the two extreme limits of density distribution, namely, (i) the uniform distribution of density ($\rho = \lambda = \text{constant}$), and (ii) the complete concentration of the mass towards the centre. The former will be referred to here as the *Maclaurin model* and the latter is of course the usual *Roche model.* We shall consider the equilibrium of these models as well, and relate their theory to the theory of polytropes. There is a consequent simplification in the theory of these configurations, and in some cases (as in the tidal problem for instance) solutions of decidedly greater accuracy than have been given so far are obtained by very elementary methods.

The following two sections are devoted to obtain the solutions for the Maclaurin and the Roche models for the three problems. The three subsequent sections consider the rotational, the tidal and the double-star problems in turn.

I. *Solutions for the Maclaurin Model*

§ 2. The Maclaurin model corresponds to a polytrope with index $n = 0$. This polytrope differs from the other polytropes in this that the density does not vanish on the boundary. If the density vanishes on the boundary, then when the configuration is distorted by a slow uniform rotation, for instance, the density in the "distorted regions" is of order v ($= \omega^2/2\pi G\lambda$), and since the volume distortion is also of order v, the mass enclosed in these regions is of order v^2, and consequently the effects arising from this mass of order v^2 can be neglected when we are working only to the first order in v. In the case of the Maclaurin model, however, since the density does not vanish on the boundary, the mass enclosed in the "distorted regions" is of order v, and we cannot therefore straightaway use the solutions obtained in I for this special case. We shall treat the problem *de novo.*

§ 3. *The General Procedure.*—The general procedure is always to determine the internal and the external potentials V and U, and making the solution for the density distribution determinate by making these potentials and their derivatives continuous on the boundary of the configuration. If $\mathbf{V}_0$ and $\mathbf{U}_0$ are the corresponding potentials for the undistorted configuration we then have

$$\left.\begin{aligned} V &= \mathbf{V}_0 + \epsilon \mathbf{V}_1, \\ U &= \mathbf{U}_0 + \epsilon \mathbf{U}_1, \end{aligned}\right\} \tag{1}$$

where $\epsilon = v$ in the rotational and the double-star problems and $\epsilon = (a/R)^3$

* "The Equilibrium of Distorted Polytropes, I," *M.N.*, **93**, 390, 1933; "The Equilibrium of Distorted Polytropes, II," *M.N.*, **93**, 449, 1933; "The Equilibrium of Distorted Polytropes, III," *M.N.*, **93**, 462, 1933. These three papers will be referred to as I, II and III in the sequel.

for the simple tidal problem. Now the boundary ξ_0 is given by an equation of the form

$$\xi_0 = \xi_1 + \epsilon\xi_2 \quad \text{(say)}. \tag{2}$$

The boundary conditions can now be expressed as

$$\left.\begin{aligned}(\mathbf{V}_0 - \mathbf{U}_0)_{\xi_1} + \epsilon\{\xi_2(\mathbf{V}_0' - \mathbf{U}_0') + (\mathbf{V}_1 - \mathbf{U}_1)\}_{\xi_1} = 0,\\ (\mathbf{V}_0' - \mathbf{U}_0')_{\xi_1} + \epsilon\{\xi_2(\mathbf{V}_0'' - \mathbf{U}_0'') + (\mathbf{V}_1' - \mathbf{U}_1')\}_{\xi_1} = 0.\end{aligned}\right\} \tag{3}$$

The terms in the different powers of ϵ must vanish separately. Hence we should have

$$(\mathbf{V}_0 - \mathbf{U}_0)_{\xi_1} = 0\,; \qquad (\mathbf{V}_0' - \mathbf{U}_0')_{\xi_1} = 0, \tag{4}$$

and

$$\left.\begin{aligned}(\mathbf{V}_1 - \mathbf{U}_1)_{\xi_1} = 0,\\ \xi_2(\mathbf{V}_0'' - \mathbf{U}_0'')_{\xi_1} + (\mathbf{V}_1' - \mathbf{U}_1')_{\xi_1} = 0.\end{aligned}\right\} \tag{5}$$

Now, in *general* for polytropes the Poisson and the Laplace equations defining $\mathbf{V}_0$ and $\mathbf{U}_0$ reduce to (as one easily verifies)

$$\left.\begin{aligned}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\mathbf{V}_0}{d\xi}\right) = -(n+1)K\lambda^{\frac{1}{n}}\theta^n,\\ \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\mathbf{U}_0}{d\xi}\right) = 0.\end{aligned}\right\} \tag{6}$$

From (6) we see that

$$\frac{d^2}{d\xi^2}(\mathbf{V}_0 - \mathbf{U}_0) + \frac{2}{\xi}\frac{d}{d\xi}(\mathbf{V}_0 - \mathbf{U}_0) + (n+1)K\lambda^{\frac{1}{n}}\theta^n = 0. \tag{7}$$

Since $\theta = 0$ for $\xi = \xi_1$ we see that $(\mathbf{V}_0'' - \mathbf{U}_0'')_{\xi_1}$ is also zero on the sphere $\xi = \xi_1$. Hence in general the conditions (5) simplify to

$$(\mathbf{V}_1 - \mathbf{U}_1)_{\xi_1} = 0\,; \qquad (\mathbf{V}_1' - \mathbf{U}_1')_{\xi_1} = 0. \tag{8}$$

It was the condition (8) which was used in the analysis in I, II and III. But the arguments which simplify (5) to (8) break down when $n = 0$, and the boundary conditions to be used for the Maclaurin model are those given in (5).

(A) The Rotational Problem :—

§ 4. Equation (5), I, continues to hold good. Let the constant value of the density ρ be λ, and put

$$P = P_c\Theta, \tag{9}$$

where P_c is the central pressure. Introduce further the variable ξ defined by

$$r = \left[\frac{P_c}{4\pi G\lambda^2}\right]^{\frac{1}{2}}\xi. \tag{10}$$

In place of (8), I, we have now

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-1+v, \tag{11}$$

where v has the usual meaning.

§ 5. *Non-rotating Configuration.*—When there is no rotation let $P=P_c\theta$. Then the differential equation for θ is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-1, \tag{12}$$

the required solution of which is

$$\theta=1-\tfrac{1}{6}\xi^2, \tag{13}$$

giving

$$\xi_1=\sqrt{6}\,; \qquad \theta_1'=-\tfrac{1}{3}\sqrt{6}. \tag{14}$$

If $\mathbf{V}_0$ and $\mathbf{U}_0$ are the potentials, then the Poisson and the Laplace equations defining them are readily found to be

$$\left.\begin{aligned}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\mathbf{V}_0}{d\xi}\right)&=-\frac{P_c}{\lambda},\\ \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\mathbf{U}_0}{d\xi}\right)&=0.\end{aligned}\right\} \tag{15}$$

For definiteness we shall assume that the zero of the potential is on the boundary of the undistorted configuration, *i.e.* on a sphere of radius ξ_1. From (15) and the boundary conditions (4) we obtain

$$\mathbf{V}_0=\frac{P_c}{\lambda}(1-\tfrac{1}{6}\xi^2)\,; \qquad \mathbf{U}_0=\frac{P_c}{\lambda}\left(-2+\frac{2\sqrt{6}}{\xi}\right). \tag{16}$$

§ 6. *Solution for the Rotating Configuration.*—To solve (11) we take as usual the following form for Θ:—

$$\Theta=\theta+v\{\psi_0(\xi)+\sum_{j=1}^{\infty}A_j\psi_j(\xi)P_j(\mu)\}+O(v^2). \tag{17}$$

The differential equations defining ψ_0 and ψ_j are

$$\left.\begin{aligned}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)&=1,\\ \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)&=\frac{j(j+1)}{\xi^2}\psi_j.\end{aligned}\right\} \tag{18}$$

Equations (18) can be integrated as they stand, and yield

$$\psi_0=\tfrac{1}{6}\xi^2\,; \qquad \psi_j=\xi^j. \tag{19}$$

The calculation of the potential proceeds exactly as in I, § 6, and yields

$$V=\frac{P_c}{\lambda}[\Theta-\tfrac{1}{6}v\xi^2(1-P_2(\mu))]. \tag{20}$$

We rewrite (20) in the form

$$V=\frac{P_c}{\lambda}\left[(1-\tfrac{1}{6}\xi^2)+v\left\{\tfrac{1}{6}\xi^2P_2(\mu)+\sum_{j=1}^{\infty}A_j\xi^jP_j(\mu)\right\}\right]. \tag{21}$$

Similarly the outer potential U is found to be

$$U=\frac{P_c}{\lambda}\left[\left(-2+\frac{2\sqrt{6}}{\xi}\right)+v\left\{C_0'+\frac{C_0}{\xi}+\sum_{j=1}^{\infty}\frac{C_j}{\xi^{j+1}}P_j(\mu)\right\}\right], \tag{22}$$

where C_0, C_0' and the C_j's are arbitrary. The boundary conditions (5) will determine these as well as the A_j's.

To proceed further we must also know the equation of the boundary in terms of the A_j's. The boundary is given by $\Theta=0$, and since Θ is given by

$$\Theta=\theta+v\left\{\tfrac{1}{6}\xi^2+\sum_{j=1}^{\infty}A_j\xi^jP_j(\mu)\right\}, \tag{23}$$

we have

$$\xi_0=\xi_1+\frac{3v}{\sqrt{6}}\left\{1+\sum_{j=1}^{\infty}A_j{\xi_1}^jP_j(\mu)\right\}. \tag{24}$$

Hence ξ_2 occurring in (2) is given by

$$\xi_2=\tfrac{1}{2}\xi_1\left\{1+\sum_{j=1}^{\infty}A_j{\xi_1}^jP_j(\mu)\right\}. \tag{25}$$

The boundary conditions (5) reduce to the following set of equations on equating the coefficients of $P_j(\mu)$:—

$$\left.\begin{aligned} C_0'+\frac{C_0}{\xi_1}&=0,\\ A_j{\xi_1}^j=\frac{C_j}{{\xi_1}^{j+1}},\quad (j\neq 2)\,;&\qquad 6A_2+1=\frac{C_2\,.\,\sqrt{6}}{36},\quad (j=2),\end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned} \frac{C_0}{{\xi_1}^2}=\tfrac{1}{2}\xi_1\,;\qquad (j-3)A_j{\xi_1}^{j-1}&=-\frac{(j+1)C_j}{{\xi_1}^{j+2}},\quad (j\neq 2)\\ -A_2\xi_1+\tfrac{1}{3}\xi_1&=-\frac{3C_2}{36},\quad (j=2).\end{aligned}\right\} \tag{27}$$

Solving these sets of equations we obtain

$$C_0=\tfrac{1}{2}{\xi_1}^3=3\sqrt{6}\,;\qquad C_0'=-\tfrac{1}{2}{\xi_1}^2=-3. \tag{28}$$

If $j\neq 2$,

$$(j-1)A_j=0, \tag{29}$$

which shows that $A_j=C_j=0$ if $j\neq 2,\ \neq 1$. A_1 is left undetermined. Going to the limit $v=0$ we see that $A_1=C_1=0$. Finally if $j=2$,

$$A_2=-\tfrac{5}{12}\,;\qquad C_2=-9\sqrt{6}. \tag{30}$$

Hence our solution is

$$\Theta=1-\tfrac{1}{6}\xi^2+v\tfrac{1}{6}\xi^2(1-\tfrac{5}{2}P_2(\mu)). \tag{31}$$

Also we note that

$$\left.\begin{aligned} V &= \frac{P_c}{\lambda}\{1 - \tfrac{1}{6}\xi^2 - \tfrac{1}{4}v\xi^2 P_2(\mu)\}, \\ U &= \frac{P_c}{\lambda}\left\{\left(-2 + \frac{2\sqrt{6}}{\xi}\right) + v\left(-3 + \frac{3\sqrt{6}}{\xi} - \frac{9\sqrt{6}}{\xi^3}P_2(\mu)\right)\right\}. \end{aligned}\right\} \tag{32}$$

By (24) the equation of the boundary is given by

$$\xi_0 = \sqrt{6}\{1 + \tfrac{1}{2}v(1 - \tfrac{5}{2}P_2(\mu))\}. \tag{33}$$

From (33) we deduce that

$$\left.\begin{aligned} &\text{Fractional Elongation at Equator} = 9/8.v, \\ &\text{Fractional Contraction at Poles} = 3/4.v, \\ &\text{Oblateness } \sigma = 15/8.v. \end{aligned}\right\} \tag{34}$$

The eccentricity e^2 of an ellipse is defined by $(a^2 - b^2)/a^2$, where a denotes the major axis and b the minor axis. When the eccentricity is small it is equal to twice the oblateness σ (to the first order). Hence from (34) we obtain

$$e^2 = \frac{15}{4}\frac{\omega^2}{2\pi G\lambda}. \tag{35}$$

§ 7. *Comparison with the Known Solution.*—The configurations of equilibrium of rotating masses of uniform density can be worked out accurately and the "evolutionary sequence"—Maclaurin spheroids, Jacobian ellipsoids—exactly followed. For the Maclaurin spheroids the exact solution is *

$$\frac{\omega^2}{2\pi G\lambda} = \frac{3 - 2e^2}{e^3}(1 - e^2)^{\frac{1}{2}} \sin^{-1}e - 3\left(\frac{1}{e^2} - 1\right), \tag{36}$$

or, when e^2 is small,

$$\frac{\omega^2}{2\pi G\lambda} = \frac{4}{15}e^2 + O(e^4), \tag{36'}$$

in agreement with (35).

§ 8. *Some Relations for the Polytrope* "n = 0."—The volume N_ω of the configuration is easily found. We have

$$N_\omega = N_0(1 + 1{\cdot}5v). \tag{37}$$

Since the density is supposed to remain constant we have clearly

$$M_\omega = M_0(1 + 1{\cdot}5v). \tag{38}$$

Again when we are comparing configurations with equal mass, we require the fractional change in the central pressure when the configuration is set rotating ; *i.e.* we have to find δP_c such that

$$M(P_c, \omega) = M(P_c - \delta P_c, 0). \tag{39}$$

* *Cf.*, for instance, J. H. Jeans, *Stellar Dynamics and Problems of Cosmogony* (1919), § 38.

By (10) M_0 is proportional to $P_c^{3/2}$. We clearly have therefore

$$\frac{\delta P_c}{P_c} = -v. \tag{40}$$

(B) The Tidal Problem :—

§ 9. Equation (11), II, continues to hold good. Introducing as in § 4 the variables ξ and Θ, we find that the equation we have to solve is

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right)=-1. \tag{41}$$

To solve (41) we assume for Θ the form *

$$\Theta=\theta+\left(\frac{\alpha}{R}\right)^3\sum_{j=1}^{\infty}A_j\psi_j(\xi)P_j(\mu)+\left(\frac{\alpha}{R}\right)^6\Phi+\cdots \tag{42}$$

The differential equation for ψ_j is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\frac{j(j+1)}{\xi^2}\psi_j. \tag{43}$$

The required solution is therefore

$$\psi_j=\xi^j. \tag{44}$$

The calculation of the potential proceeds exactly as in II, § 4, and yields

$$V=\frac{P_c}{\lambda}\left[\theta+\left(\frac{\alpha}{R}\right)^3\left\{\sum_{j=1}^{\infty}A_j\xi^jP_j(\mu)-\frac{GM'}{R}\frac{\lambda}{P_c}\sum_{j=2}^{4}\left(\frac{\alpha}{R}\right)^{j-3}\xi^jP_j(\mu)\right\}\right],$$

which we rewrite in the form

$$V=\frac{P_c}{\lambda}\left[(1-\tfrac{1}{6}\xi^2)+\left(\frac{\alpha}{R}\right)^3\left\{\sum_{j=1}^{\infty}A_j\xi^jP_j(\mu)-2\sqrt{6}\,.\,\frac{M'}{M}\sum_{j=2}^{4}\left(\frac{\alpha}{R}\right)^{j-2}\xi^jP_j(\mu)\right\}\right]. \tag{45}$$

Similarly

$$U=\frac{P_c}{\lambda}\left[\left(-2+\frac{2\sqrt{6}}{\xi}\right)+\left(\frac{\alpha}{R}\right)^3\sum_{j=1}^{\infty}\frac{C_j}{\xi^{j+1}}P_j(\mu)\right]. \tag{46}$$

By (42) the equation of the boundary takes the form

$$\xi_0=\xi_1+\left(\frac{\alpha}{R}\right)^3\,.\,\frac{3}{\sqrt{6}}\sum_{j=1}^{\infty}A_j\,.\,\xi_1{}^jP_j(\mu). \tag{47}$$

* The differential equation (41) contains no reference to any small parameter, and it is not necessary in fact to anticipate our solution to be a series in powers of $(\alpha/R)^3$ as we have done in (42). Actually the $(\alpha/R)^3$ in front of the second term could have been absorbed in the A_j's, and we could assume a solution (as in II) of the form $\theta+\Psi$ and neglect squares of Ψ. The result will be the same, and as such (42) is the really more convenient form to work with.

The boundary conditions (5) resolve into the following sets of simultaneous equations:—

$$\left.\begin{aligned} A_j\xi_1{}^j &= \frac{C_j}{\xi_1{}^{j+1}}, \\ (j-3)A_j\xi_1{}^{j-1} &= -(j+1)\cdot\frac{C_j}{\xi_1{}^{j+2}}, \end{aligned}\right\} \tag{48}$$

for $j \neq 2, \neq 3, \neq 4$, and

$$\left.\begin{aligned} A_j\xi_1{}^j - 2\sqrt{6}\cdot\frac{M'}{M}\left(\frac{\alpha}{R}\right)^{j-2}\cdot\xi_1{}^j &= \frac{C_j}{\xi_1{}^{j+1}}, \\ (j-3)A_j\xi_1{}^{j-1} - 2\sqrt{6}\cdot\frac{M'}{M}\left(\frac{\alpha}{R}\right)^{j-2}\cdot j\cdot\xi_1{}^{j-1} &= -(j+1)\frac{C_j}{\xi_1{}^{j+2}}, \end{aligned}\right\} \tag{49}$$

for $j = 2$, 3 or 4.

Solving we find

$$\left.\begin{aligned} A_j &= C_j = 0, \qquad j \neq 2, \neq 3, \neq 4, \\ A_j &= \frac{(2j+1)}{2(j-1)}\cdot\xi_1{}^2\,|\,\theta_1'\,|\cdot\frac{M'}{M}\left(\frac{\alpha}{R}\right)^{j-2}, \\ C_j &= \frac{3}{2(j-1)}\cdot\xi_1{}^{2j+3}\,|\,\theta_1'\,|\cdot\frac{M'}{M}\left(\frac{\alpha}{R}\right)^{j-2}, \end{aligned}\right\} \tag{50}$$

for $j = 2$, 3, 4.

Hence our solution is given by

$$\Theta = 1 - \tfrac{1}{6}\xi^2 + 2\frac{M'}{M}\cdot\sum_{j=2}^{4}\nu^{j+1}\cdot\frac{2j+1}{2(j-1)}\cdot\frac{\xi^j}{6^{j/2}}P_j(\mu), \tag{51}$$

where $\nu(=\alpha\xi_1/R)$ has the same meaning as in II and III.

The equation of the boundary ξ_0 can now be expressed in the "standard form" (*cf.* II, equation (35)),

$$\frac{\xi_0 - \xi_1}{\xi_1} = \frac{M'}{M}\sum_{j=2}^{4}\Delta_j\cdot\nu^{j+1}\cdot P_j(\mu), \tag{52}$$

if we *define* for the polytrope "$n = 0$" the Δ_j's in the following manner:—

$$\Delta_j = \frac{2j+1}{2(j-1)}, \qquad j = 2, 3, 4. \tag{53}$$

This *definition* for the Δ_j's has *always* to be used whenever we are dealing with the Maclaurin model. (For the other polytropes the general definition of the Δ_j's (equation (29), II) has to be used.)

Explicitly written out, (52) takes the form

$$\frac{\xi_0 - \xi_1}{\xi_1} = \frac{M'}{M}\cdot\nu^3[2{\cdot}5P_2(\mu) + 1{\cdot}75\nu P_3(\mu) + 1{\cdot}5\nu^2P_4(\mu)]. \tag{54}$$

When the secondary is so far away that we could consider the configuration as a spheroid, then we have to retain only the first term in (54). We then have

$$\left.\begin{aligned}\text{Fractional Elongation at Poles} &= 2{\cdot}5\nu^3 \cdot \frac{M'}{M},\\ \text{Fractional Contraction at Equator} &= 1{\cdot}25\nu^3 \cdot \frac{M'}{M}.\end{aligned}\right\} \tag{55}$$

Also,

$$\text{Eccentricity, } e^2 = 7{\cdot}5\nu^3\frac{M'}{M} + E(\nu^4), \tag{56}$$

where by $E(\nu^4)$ we mean that the "error term" is of $O(\nu^4)$.

§ 10. *Comparison with the Known Solution.*—If the tidal field is *strictly* of the form (*cf.* equation (6), II)

$$\frac{M'r^2}{R^3}P_2(\cos\theta), \tag{57}$$

then the tidal problem admits of exact solution and is given by (*cf.* J. H. Jeans, *loc. cit.*, § 49)

$$\frac{M'}{\pi R^3\lambda} = \frac{1-e^2}{e^3}\log\left(\frac{1+e}{1-e}\right) - \frac{6(1-e^2)}{e^2(3-e^2)}. \tag{58}$$

For small e^2 we find

$$\frac{M'}{\pi R^3\lambda} = \frac{8}{45}e^2 + O(e^4), \tag{59}$$

or

$$e^2 = 7{\cdot}5\frac{M'}{M}\cdot\nu^3 + O(\nu^6). \tag{60}$$

(60) agrees with (56) in the principal term, but the order of the omitted terms in (56) and (60) do not agree. This difference arises because in obtaining the solution (54) we have not merely used the first term in the expansion (6), II, as is done in obtaining the "exact solution" (58).

Actually consistent with the approximation of treating the secondary as a mass point, we should take into account not only the term in $P_2(\mu)$ but also the two subsequent terms involving $P_3(\mu)$ and $P_4(\mu)$ which occur in the expansion for the outer potential of the secondary. These terms arising from the nearness of the secondary distort the configuration from the true spheroidal form, and these effects are of $O(\nu^4)$.—On the whole the solution as expressed by (54) is preferable to the "exact solution" (58).

§ 11. In II, § 8, we considered the angle at which the furrow in the external shape develops. Since the equation of the boundary (52) is of the standard form, the analysis contained there applies to the Maclaurin model as well. Thus

$$\chi = \tfrac{1}{2}\frac{\Delta_3}{\Delta_2}\nu + O(\nu^3),$$

or by (53)

$$\chi = 0{\cdot}35\nu + O(\nu^3). \tag{61}$$

(C) The Double-star Problem :—

§ 12. There is no need to go into details to obtain the solution. We can write this at once by an application of the superposition theorem (III, § 7), which clearly holds good for the Maclaurin model as well. Thus by (31) and (51) we have

$$\Theta = \theta + v \,.\, \tfrac{1}{6}\xi^2(1 - \tfrac{5}{2}P_2(\mu)) + 2\frac{M'}{M}\sum_{j=2}^{4} \nu^{j+1} \,.\, \frac{2j+1}{2(j-1)} \,.\, \frac{\xi^j}{6^{j/2}} \,.\, P_j\,(\sin\theta\cos\phi), \quad (62)$$

or differently as

$$\Theta = 1 - \tfrac{1}{6}\xi^2 + \tfrac{2}{3}\left(1 + \frac{M'}{M}\right) \,.\, \nu^3 \,.\, \tfrac{1}{6}\xi^2(1 - \tfrac{5}{2}P_2(\mu)) + 2\frac{M'}{M}\sum_{j=2}^{4} \nu^{j+1} \,.\, \frac{2j+1}{2(j-1)} \,.\, \frac{\xi^j}{6^{j/2}} P_j(\sin\theta\cos\phi). \quad (63)$$

The equation of the boundary is

$$\frac{\xi_0 - \xi_1}{\xi_1} = \tfrac{1}{3}\left(1 + \frac{M'}{M}\right)\nu^3 - \tfrac{5}{6}\left(1 + \frac{M'}{M}\right)\nu^3 P_2(\cos\theta) + \frac{M'}{M}\{2{\cdot}5\nu^3 P_2(\sin\theta\cos\phi) + 1{\cdot}75\nu^4 P_3(\sin\theta\cos\phi) + 1{\cdot}5\nu^5 P_4(\sin\theta\cos\phi)\}. \quad (64)$$

Thus there is an expansion of amount

$$\tfrac{1}{3}\left(1 + \frac{M'}{M}\right) \,.\, \nu^3, \quad (65)$$

and superposed on this are the "ellipticity" and "harmonic" terms. To study the *deviations* from a spherical volume we can define as in III, equation (39),

$$\sigma(\theta, \phi) = \frac{\xi_0 - \xi_1}{\xi_1} - \tfrac{1}{3}\left(1 + \frac{M'}{M}\right) \,.\, \nu^3. \quad (66)$$

We readily verify that we can express $\sigma(\theta, \phi)$ in the form:

$$\sigma(\theta, \phi) = -\tfrac{1}{3}\left(1 + \frac{M'}{M}\right) \,.\, \Delta_2 \,.\, \nu^3 P_2(\mu) + \frac{M'}{M} \,.\, \sum_{j=2}^{4} \Delta_j \,.\, \nu^{j+1} \,.\, P_j(\sin\theta\cos\phi), \quad (67)$$

with the Δ_j's defined by (53). (67) is of the same form as (39), III, and the subsequent discussions in III apply to the Maclaurin model as well.

We may note explicitly that to the order of accuracy where the equilibrium configurations are ellipsoids the eccentricities in the three principal planes are (*cf.* equation (43′), III)

$$\left.\begin{aligned} e_e^{\,2} &= 7{\cdot}5\frac{M'}{M} \,.\, \nu^3, \\ e_m^{\,2} &= 2{\cdot}5\left(1 + 4\frac{M'}{M}\right) \,.\, \nu^3, \\ e_d^{\,2} &= 2{\cdot}5\left(1 + \frac{M'}{M}\right) \,.\, \nu^3. \end{aligned}\right\} \quad (68)$$

Further

$$\left.\begin{aligned}\chi_m &= \frac{M'}{M+4M'} \times 1{\cdot}05\nu,\\ \chi_e &= \qquad\qquad 0{\cdot}35\nu.\end{aligned}\right\} \tag{69}$$

II. *Solutions for the Roche Model*

§ 13. The Roche model corresponds to the polytrope $n=5$. Since, however, the actual Emden functions employed are those which take the value unity at the origin, the configuration extends to infinity, and one has therefore to be rather careful in performing the limiting process.

§ 14. *The Rotational Problem.*—Now the equation of the boundary for a polytrope of index $n \neq 5$ is given by equation (38), I:

$$\xi_0 = \xi_1 + \frac{v}{|\theta_1'|}\left[\psi_0(\xi_1) - \frac{5}{6}\frac{\xi_1^2\psi_2(\xi_1)}{3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)}P_2(\mu)\right]. \tag{70}$$

When we are comparing the distortions as a function of the density distribution we keep the mean density constant. Instead of v we introduce therefore the quantity ζ defined by (*cf.* equation (46), I)

$$\zeta = \frac{\omega^2}{2\pi G\rho_m} = \frac{\omega^2}{2\pi G\lambda}\cdot\frac{\xi_1}{3|\theta_1'|}, \tag{71}$$

or

$$v = \frac{3|\theta_1'|}{\xi_1}\cdot\zeta. \tag{71'}$$

Introducing this in (70) we can rewrite the equation of the boundary in the following way :—

$$\frac{\xi_0-\xi_1}{\xi_1} = \zeta\left[\frac{3\psi_0(\xi_1)}{\xi_1^2} - \tfrac{1}{2}\Delta_2\cdot P_2(\mu)\right]. \tag{72}$$

Now ψ_0 and ψ_2 satisfy the differential equations :

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1, \tag{73}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(\frac{6}{\xi^2} - n\theta^{n-1}\right)\psi_2. \tag{74}$$

Now, as is known, when the polytropic index n approaches 5 the radius ξ_1 tends to infinity and at the same time θ is very nearly zero long before it actually crosses the ξ-axis at ξ_1. Hence as $n \to 5$ the differential equation satisfied by ψ_0 will approximate to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = 1, \tag{73'}$$

the solution of which is

$$\psi_0 = \tfrac{1}{6}\xi^2. \tag{73''}$$

Hence in (72) we have to put

$$\frac{3\psi_0(\xi_1)}{\xi_1^2} = \tfrac{1}{2}, \qquad \text{for} \qquad n = 5. \tag{75}$$

Similarly from (74) we see that as $n \to 5$ the differential equation for ψ_2 for a long way from the boundary is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \frac{6\psi_2}{\xi^2}, \tag{74'}$$

the solution of which is of the form:

$$\psi_2 = B\xi^2, \tag{74''}$$

the constant B depending on the whole march of θ. But Δ_2 is found to be unity independent of B. Hence we have to put

$$\Delta_2 = 1 \tag{76}$$

for $n = 5$. Hence our solution is

$$\frac{\xi_0 - \xi_1}{\xi_1} = \tfrac{1}{2}\zeta(1 - P_2(\mu)) = \tfrac{3}{4}\zeta \sin^2\theta. \tag{77}$$

The solution (77) is obtained by decidedly non-rigorous mathematical arguments, but it would be very difficult to make the arguments more exact, as it would require, for instance, the way, $\xi_1 \to \infty$ as $n \to 5$, and similar theorems where we regard the Emden functions as a function of the two variables ξ and n. It is, however, an easy matter to verify that the solution (77) is consistent with the usual treatment of the Roche model which is the following :—

If we regard the whole mass M as being concentrated at the origin, then the sum of the gravitational and the centrifugal potentials is given by

$$W = \frac{GM}{r} + \tfrac{1}{2}\omega^2(x^2 + y^2). \tag{78}$$

The surfaces $W =$ constant are equipotentials, and if there is a configuration of equilibrium we should have a closed equipotential, which we will always have when ω is smaller than a certain critical value. The boundary of the configuration must also be an equipotential, and we select from the infinity of the equipotentials, $W =$ constant, the one which encloses a volume just adequate to contain the whole amount of matter.

Now if R_1 is the radius of the original configuration, then the equation of the boundary given by (77) is of the form:

$$R_0 = R_1(1 + \tfrac{3}{4}\zeta \sin^2\theta). \tag{79}$$

Hence

$$\frac{GM}{R_0} = \frac{GM}{R_1}(1 - \tfrac{3}{4}\zeta \sin^2\theta),$$

or substituting the zero order approximation $M = 4/3 \cdot \pi R_1^3 \rho_{\text{mean}}$ for the term in ζ we find that

$$\frac{GM}{R_0} = \frac{GM}{R_1} - \tfrac{1}{2} R_1^2 \omega^2 \sin^2 \theta. \tag{80}$$

Again,

$$\tfrac{1}{2}\omega^2(x^2+y^2) = \tfrac{1}{2} R_1^2 \omega^2 \sin^2 \theta. \tag{80'}$$

Hence

$$W(R_0) = \frac{GM}{R_1} = \text{constant}. \tag{81}$$

Thus the surface (79) is an equipotential and encloses a volume just sufficient to contain the whole amount of matter, thus verifying our solution (77). This consistency of (77) with the correct solution of the Roche model makes it very likely—if not certain—that the type of arguments used to derive it will in any case give the correct results when we want to consider the polytrope $n = 5$.

Finally we note that for the Roche model

$$\left.\begin{aligned} &\text{Fractional Elongation at Equator} = 0{\cdot}75\zeta, \\ &\text{Fractional Contraction at Poles} \quad = 0, \\ &\text{Oblateness } \sigma \qquad\qquad\quad = 0{\cdot}75\zeta. \end{aligned}\right\} \tag{82}$$

§ 15. *The Tidal and the Double-star Problems.*—To study the external shape for the tidally distorted configuration we need only the Δ_j's. By means of arguments similar to those used to derive (76) we infer that for the Roche model we have to set all the Δ_j's to be equal to unity. The numerical results in II (Table VI) convinces us of the correctness of this procedure (*cf.* the remarks in italics at the end of § 7 in II). Thus for the Roche model the equation of the boundary is given by (*cf.* equation (45), II)

$$\xi_0 = \xi_1 \left\{ 1 + \frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot P_j(\mu) \right\}. \tag{83}$$

By the methods used in the second half of § 14 we can easily verify that the solution (83) is consistent with the usual treatment of the Roche model.

The angle χ at which the furrow develops in the external shape for the Roche model is given by

$$\chi = 0{\cdot}5\nu + O(\nu^3). \tag{84}$$

The double-star problem requires no special remarks, but we may note that to the order of accuracy, where the equilibrium configurations are ellipsoids, the eccentricities in the three principal planes are given by

$$\left.\begin{aligned} e_e^2 &= 3\frac{M'}{M} \cdot \nu^3, \\ e_d^2 &= \left(1 + \frac{M'}{M}\right) \cdot \nu^3, \\ e_m^2 &= \left(1 + \frac{4M'}{M}\right) \cdot \nu^3. \end{aligned}\right\} \tag{85}$$

Further

$$\left.\begin{aligned}\chi_m &= 1{\cdot}5\frac{M'}{M+4M'}\nu + O(\nu^3),\\ \chi_e &= 0{\cdot}5\nu \qquad\qquad + O(\nu^3).\end{aligned}\right\} \tag{86}$$

III. *The Rotational Problem*

§ 16. *Results for the Polytrope* "n = 3·5."—The necessary numerical integrations for the polytropes $n = 1$, 1·5, 2, 3 and 4 have been made in I. However, to draw unambiguous conclusions regarding the dependence of the rotational distortions on the density distribution it was found necessary to have the results for one more polytrope with $3 < n < 4$. The functions ψ_0 and ψ_2 for $n = 3{\cdot}5$ are tabulated in the Appendix II.

The values of ψ_0 and ψ_2 and their derivatives at ξ_1 are

$$\left.\begin{aligned}&\xi_1 = 9{\cdot}5358,\\ &\psi_0(\xi_1) = 12{\cdot}4466, \qquad \psi_2(\xi_1) = 20{\cdot}1390,\\ &\psi_0'(\xi_1) = \ 2{\cdot}9611, \qquad \psi_2'(\xi_1) = \ 4{\cdot}0706.\end{aligned}\right\} \tag{87}$$

The numerical forms for the various formulæ may be noted :

$$\Theta = \theta + v[\psi_0(\xi) - 0{\cdot}7636\psi_2(\xi)P_2(\mu)], \tag{88}$$

$$\xi_0 = \xi_1 + 48{\cdot}098v[12{\cdot}4466 - 15{\cdot}379P_2(\mu)]. \tag{89}$$

$$\left.\begin{aligned}&\text{Fractional Elongation at Equator} = 101{\cdot}57\ v,\\ &\text{Fractional Contraction at Poles} \ = \ 14{\cdot}789v,\\ &\text{Oblateness, } \sigma \qquad\qquad\quad = 116{\cdot}36\ v.\end{aligned}\right\} \tag{90}$$

$$\left.\begin{aligned}M_\omega &= M_0[1 + 10{\cdot}463v],\\ N_\omega &= N_0[1 + 188{\cdot}34v],\\ \rho_c &= \rho_m \times 152{\cdot}884[1 + 177{\cdot}88v].\end{aligned}\right\} \tag{91}$$

§ 17. *Distortions of the External Shape as a Function of the Polytropic Index.*—In I all the results were expressed in terms of v as it is this small parameter which appears explicitly in the differential equation for Θ. But when we want to compare the distortions in the different polytropes we should not compare configurations with the same central density (λ), since polytropes with differing n but equal λ are characterised by different masses and different radii. To compare polytropes having the same mass and the same radius we must compare configurations with the same mean density. Hence we must rewrite all our equations in terms of ζ (equation (71)).

Now the equation of the boundary can be expressed in the form (equation (72)) :

$$\frac{\xi_0 - \xi_1}{\xi_1} = \zeta\left[\frac{3\psi_0(\xi_1)}{{\xi_1}^2} - \tfrac{1}{2}\Delta_2 P_2(\mu)\right]. \tag{92}$$

From (92) we obtain the results summarised in Table I.

TABLE I

Geometry of the Boundary

n	General Expansion Term $\frac{3\psi_0(\xi_1)}{\xi_1^2}\cdot\zeta^{-1}$	Fractional Elongation at Equator $\times\zeta^{-1}$	Fractional Contraction at Poles $\times\zeta^{-1}$	Ratio of Elongation to Contraction	Oblateness, $\sigma\times\zeta^{-1}$
0	0·5	1·125	0·75	1·5	1·875
1	0·3040	0·6839	0·4560	1·5000	1·1399
1·5	0·2908	0·6131	0·3538	1·7329	0·9669
2	0·3032	0·5903	0·2709	2·1792	0·8612
3	0·3682	0·6254	0·1462	4·2762	0·7716
3·5	0·4106	0·6643	0·0967	6·868	0·7611
4	0·4488	0·6995	0·0525	13·31	0·7520
5	0·5	0·75	0	∞	0·75

The results of Table I are graphically illustrated in figs. 1 and 2. We notice the following features :—

(i) The fractional elongation at the equator as a function of n has a *minimum* somewhere in the region $n=2$.

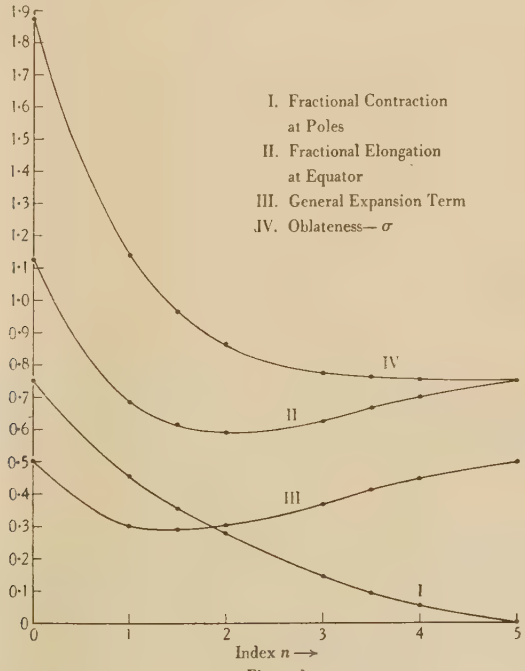

Figure 1.

(ii) The general expansion term also passes through a minimum, but for a smaller value of n.

(iii) The fractional contraction at the poles monotonically tends to zero as n tends to 5.

(iv) The ratio of the elongation at the equator to the contraction at the poles tends to infinity as $n \to 5$.* Also, this ratio has *exactly* (to the first order in ζ) the value 1·5 for $n = 0$. By numerical calculations this ratio is found to be 1·5000 for $n = 1$. This exact coincidence may be an accident, but it is clear that this ratio has a minimum for $0 < n < 1$.

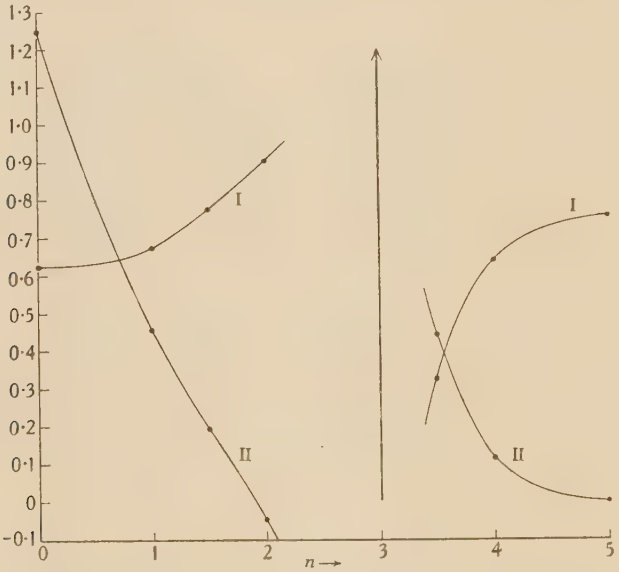

Figure 1 *a*.

The Fractional Changes in Length as a Configuration of a Given Mass is set Rotating

I. At Equator

II. At Poles

Further from (92) we see that the oblateness of the boundary of the configuration is given by

$$\sigma(\xi_0) = 0{\cdot}75\Delta_2\zeta. \tag{92'}$$

In the inner parts of the configuration strata of equal density are also characterised by ellipticities different from but *less* than that given by (92'). The oblateness for a specified value of ξ is easily found to be (*cf.* equation (39), I)

$$\sigma(\xi) = -\frac{5}{4}\frac{v}{\theta'}\frac{\xi\psi_2(\xi)}{3\psi_2(\xi) + \xi\psi_2'(\xi)}.$$

When $\xi \to 0$ we have the limiting forms

$$\theta' = -\tfrac{1}{3}\xi + O(\xi^3),$$
$$\psi_2(\xi) = \xi^2 + O(\xi^4),$$
$$\psi_2'(\xi) = 2\xi + O(\xi^3).$$

* This fact, combined with the theorem stated in III, § 9, shows that if a component of a double-star system is a polytrope with $n = 5$, then there can *never* be a contraction in its equatorial plane however large the ratio M'/M may be.

Hence

$$\sigma(\xi) \to 0{\cdot}75v \quad \text{as} \quad \xi \to 0. \tag{92''}$$

By (92′) and (92″) we see that

$$\frac{\sigma(\xi_0)}{\sigma(0)} = \Delta_2 \cdot \frac{\zeta}{v} = \Delta_2 \cdot \frac{\rho_{\text{central}}}{\rho_{\text{mean}}}. \tag{92'''}$$

We easily verify that (92‴) as it stands is also valid for the Maclaurin model when we consider the oblateness of strata of equal pressure (*i.e.*) of equal Θ.

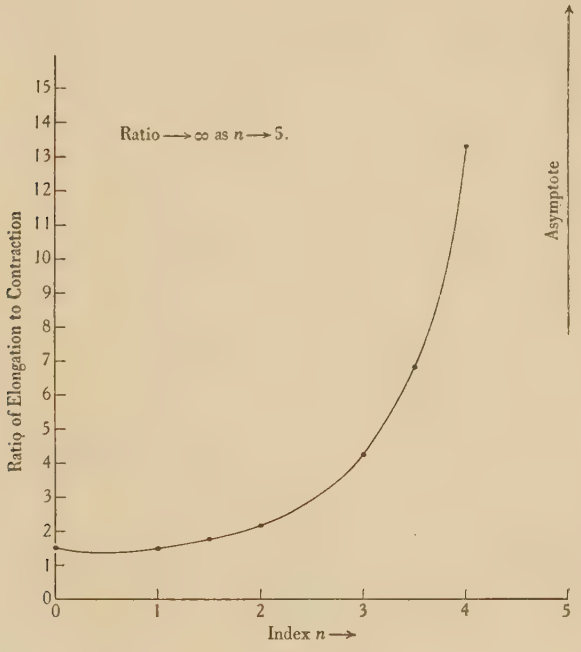

Figure 2.

§ 18. *The Oblateness as a Function of the Central Density.*—The quantity of interest in the rotational problem is the oblateness. In fig. 1 this was plotted against the polytropic index. But to obtain the qualitative nature of the dependence of the distortions on the march of the density distribution we shall plot the oblateness against the logarithm of the central density, regarding the mean density as unity (fig. 3).

Now, for the polytropic model the oblateness must be a function only of the central density, since for the polytropes the specification of the central density uniquely determines the density distribution, as the distribution function "θ" would have to be a solution of Emden's equation for some index n (and hence characterised by a unique ratio between the central and the mean densities).

Actually, however, the oblateness must be a function of the complete march of the density distribution. However that may be, we expect that

the oblateness curve in fig. 3 gives us the general nature of its dependence on the march of the density distribution. For the larger the central density, the larger is the concentration of the mass towards the centre, and conversely. Though the density distribution is not unambiguously determined by the specification of the central density, it is clear, however, that since the density would have to decrease monotonically from the centre outwards, the deviations from two different density distributions, which enclose in a prescribed spherical volume a prescribed mass and at the same time are characterised by equal densities and density gradients $\partial\rho_c/\partial r(=0)$ at the centre, cannot be

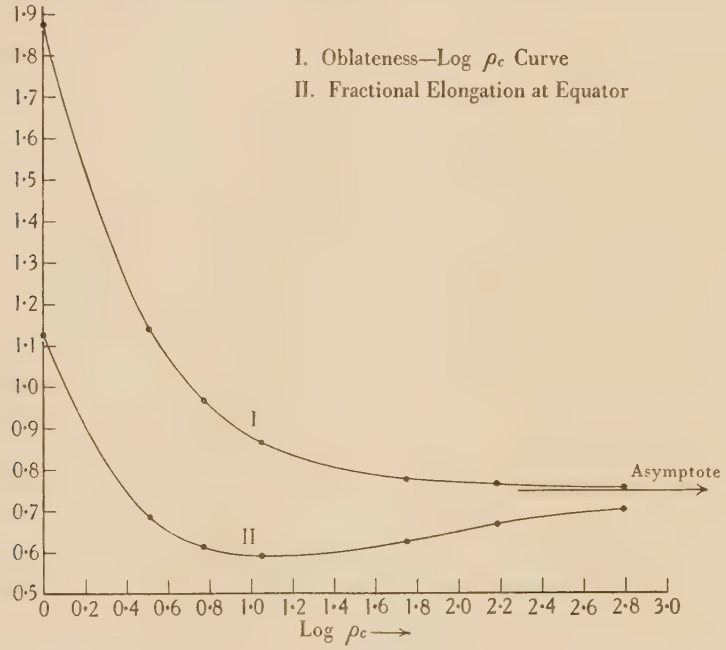

Figure 3.

very large—in any case cannot be large enough to alter the nature of the dependence of the distortions on the density distribution we have found by confining our attention to the polytropes.

As an illustration of how we can use the results obtained to deduce results of practical importance we shall attempt to estimate the central densities for the various planets for which there exist reliable information.

§ 19. *An Application to the Figure of the Earth.*—The *ellipticity* ϵ of a rotating configuration is defined as $(a-b)/a$, where a and b are the major and the minor axes of the spheroid of equilibrium. To the order of accuracy to which we are working, the ellipticity ϵ equals the quantity we have called oblateness and denoted by σ. Now for all polytropes we can express ϵ as

$$\epsilon = C \times \frac{\omega^2}{2\pi G\rho_{\text{mean}}}, \tag{93}$$

where C is a constant tabulated in the last column of Table I. Now the

value of the surface gravity g is, to the zero order approximation, given by

$$g = \tfrac{4}{3}\pi a G \rho_{\text{mean}}. \tag{94}$$

From (93) and (94) we deduce that

$$\epsilon = \tfrac{2}{3} C \cdot \frac{\omega^2 a}{g}. \tag{95}$$

The usual notation of the subject is to denote by m the quantity $\omega^2 a/g$, which is simply the ratio of the centrifugal force of the "planet's" rotation at the equator to the mean pure gravity.

Now for the Earth G. H. Darwin * gave $m = (289{\cdot}66)^{-1}$ or $m = 0{\cdot}003452$. Hence for a configuration having an "m" equal to that of the Earth we have

$$\epsilon = 0{\cdot}002301 \times C. \tag{96}$$

From the above equation the following table was constructed:—

TABLE II

ϵ for Configurations having $m = 0{\cdot}003452$

n	0	1	1·5	2	3	5
ρ_c	1	3·290	5·991	11·403	54·182	∞
$10^3 \times \epsilon$	4·314	2·623	2·225	1·982	1·776	1·726

Actually the observed ellipticity is

$$\epsilon = \tfrac{1}{297} = 3{\cdot}367 \times 10^{-3}. \tag{97}$$

By interpolation among the values given in Table II a value of ρ_c for $\epsilon = 0{\cdot}003367$ was found to be 1·8. It is a matter of doubt as to the amount of extrapolation that is being made if we actually did identify $\rho_c = 1{\cdot}8$ as indicating the kind of central densities that are to be found inside the Earth. In any case a crude estimate can be made this way.

The mean density of the Earth is 5·5 grm.-cm.$^{-3}$. Hence the estimated central density is

$$\rho_c = 1{\cdot}8 \times 5{\cdot}5 = 9{\cdot}9 \text{ grm.-cm.}^{-3}. \tag{98}$$

This agrees very well with the usual estimates for the internal densities in the Earth.†

§ 20. *Applications to other Planets.*—A quantity which is useful in this connection is $5m/2\epsilon$. By (95)

$$\frac{5m}{2\epsilon} = \frac{15}{4} \times C^{-1}. \tag{99}$$

* *Scientific Papers*, 3, 57–68.

† Compare H. Jeffreys, *The Earth* (Cambridge, 1929), pp. 218–220, where estimates ranging from 8 to 12 grm.-cm.$^{-3}$ are given.

In Table III the values for $5m/2\epsilon$ for the different polytropes are collected.

TABLE III

"$5m/2\epsilon$"

n	0	1	1·5	2	3	4	5
$5m/2\epsilon$	2	3·290	3·878	4·355	4·860	4·986	5

It is of interest to note that $5m/2\epsilon$ *must always lie between the extreme limits* 2 *and* 5 :

$$2 \leqslant \frac{5m}{2\epsilon} < 5. \tag{100}$$

The existence of an *absolute* upper limit (namely 5) to this ratio does not seem to have been noticed before. The existence of the lower limit was, however, first recognised by Laplace.

In fig. 4*a* the results of Table III are plotted. The curve increases rather sharply in the range $0 < n < 2$. In fig. 4*b* $5m/2\epsilon$ is plotted against

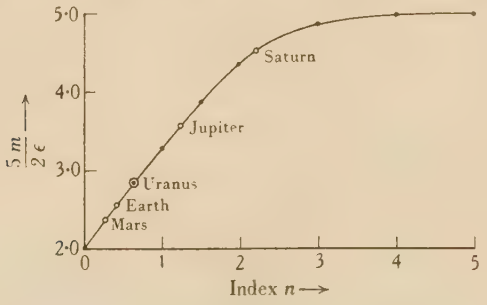

Figure 4 *a*.

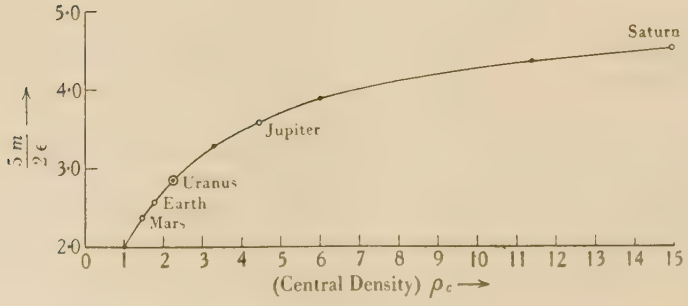

Figure 4 *b*.

ρ_c for the interesting part of the curve. If the value of $5m/2\epsilon$ for a rotating configuration is known we can estimate the central densities from this curve.

This was done for those planets for which we have reliable data.* The results are summarised in Table IV.

TABLE IV

Estimates of the Central Densities for some Planets

Planet	Period of Rotation	ϵ^{-1}	Mean Density (Density of Earth as Unity)	$5m/2\epsilon$	ρ_c/ρ_m	ρ_c in grm.-cm.$^{-3}$
	h					
Mars	24·6	190	0·65	2·38	1·5	5·3
Jupiter	9·8	16	0·23	3·57	4·45	5·6
Saturn	10·2	11	0·115	4·53	~15	~9·5
Uranus	10·8	12 – 15	0·2	2·5 – 3·2	1·6 – 3·05	3·3 – 1·8

The values of the "ρ_c/ρ_m" ratio were read off from the curve in fig. 4*b*. The actual estimates for the central densities are given in the last column. Though much significance is not to be attached to the precise values given in the last column, yet the estimates are in no way unreasonable. Jupiter is of special interest inasmuch as the ratio ρ_c/ρ_m for this planet is rather large compared with the Earth. This, however, confirms an earlier suggestion of Darwin. He says: "The surface density of Jupiter is far less than the mean, and is it not possible that Jupiter may still be in a semi-nebulous condition?" Darwin further conjectured that "the like may be true of Saturn." But at that time there was no reliable information regarding $5m/2\epsilon$, and Darwin gives values for this ratio ranging from 2·7 to 4·25. The more reliable data available now indicates that ρ_c/ρ_m is the largest for this planet.† In agreement as this is with the still gaseous nature of Saturn, it seems possible that it may have a density distribution similar to that of an Emden polytrope with index $n = 2{\cdot}2$.

§ 21. *Comparison of Configurations with Equal Mass.*—The "mass-relation" for two gas configurations with equal central densities—one rotating with an angular velocity ω and the other non-rotating—is (*cf.* equation (42), I)

$$M_\omega = M_0\left[1 + \zeta\left(1 - \frac{3\psi_0'(\xi_1)}{\xi_1}\right)\right]. \qquad (101)$$

The coefficients of ζ in the above expression for different values of n are tabulated in the second column of Table V. From (101) we easily derive

* For the observational data, see Russell, Dugan and Stewart, *Astronomy*, **1**, Appendix, Table IV, 1926; also *Handbuch der Astrophysik*, 4, 359, 1929.

† The conclusions drawn here from an application of the theory of polytropes appear to be in general agreement with some deductions of Jeffreys regarding Saturn from quite different considerations (*cf.* H. Jeffreys, *M.N.*, 84, 534, 1924).

that the fractional change in the central density which takes place when a gas sphere of a given mass is set rotating is (*cf.* equation (61), I)

$$\frac{\delta\lambda}{\lambda} = -\frac{2n}{3-n}\left(1 - \frac{3\psi_0'(\xi_1)}{\xi_1}\right) \cdot \zeta. \tag{102}$$

Another quantity which is more fundamental than $\delta\lambda/\lambda$ is the fractional change in the central pressure $(\delta P_c/P_c)$ when a gas sphere is set rotating. From (102) we easily derive that

$$\frac{\delta P_c}{P_c} = -\frac{2(n+1)}{3-n}\left(1 - \frac{3\psi_0'(\xi_1)}{\xi_1}\right) \cdot \zeta. \tag{103}$$

For $n=0$ the result is given in (40). Precise values of (102) and (103) can be obtained from the third and the fourth columns of Table V.

When we are comparing configurations of the same mass, then ξ_0 and ξ_1 do *not* represent the boundary of the two configurations on the same scale. If $r\ (\omega, \mu)$ and r_0 represent the actual boundaries for the rotating and the non-rotating configurations, then

$$r(\omega, \mu) = \left(\alpha + \frac{\partial\alpha}{\partial\lambda}\delta\lambda\right)\xi_0, \qquad \left(\alpha = \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{\frac{1}{2}}\right), \tag{104}$$

α being the scale in which ξ represents the radius and $\delta\lambda$ is equal to that given by (102). From (104) we easily deduce that

$$\frac{r(\omega, \mu) - r_0}{r_0} = \zeta\left[\frac{3\psi_0(\xi_1)}{{\xi_1}^2} + \frac{n-1}{3-n}\left(1 - \frac{3\psi_0'(\xi_1)}{\xi_1}\right) - \tfrac{1}{2}\Delta_2 P_2(\mu)\right].$$

The case $n=0$ has to be treated separately. We find that

$$\frac{r(\omega, \mu) - r_0}{r_0} = -\tfrac{5}{4}\zeta P_2(\mu). \tag{105}$$

From (105) we can calculate the fractional changes in lengths that take place

TABLE V

n	$\frac{M_\omega - M_0}{M_0}\cdot\zeta^{-1}$ (Equation (101))	$\frac{\delta\lambda}{\lambda}\cdot\zeta^{-1}$	$\frac{\delta P_c}{P_c}\cdot\zeta^{-1}$	$\frac{r(\omega, -\frac{1}{2}) - r_0}{r_0}\cdot\zeta^{-1}$, Elongation at Equator	$\frac{-r(\omega, 1) + r_0}{r_0}\cdot\zeta^{-1}$, Contraction at Poles
0	1·5	...	−1	0·625	1·25
1	0·6960 $\left(=1-\frac{3}{\pi^2}\right)$	−0·6960	−1·3921	0·6839	0·4560
1·5	0·4775	−0·9550	−1·5916	0·7723	0·1947
2	0·3135	−1·2539	−1·8809	0·9038	−0·0426
3	0·1130	...	...	...	...
3·5	0·06842	+0·9579	+1·2316	0·3222	0·4389
4	0·02188	0·1750	0·2188	0·6339	0·1182
5	0	0	0	0·75	0

at the equator and at the poles when a gas sphere of a given mass is set rotating.* The results are given in the fifth and the sixth columns in Table V and are further graphically illustrated in fig. 1*a* (p. 554).

Considering first the fractional change in length that occurs at the equator we see that there can actually be a *contraction* provided n is sufficiently near 3. This would happen, for instance, for a polytrope with $n = 3{\cdot}1$. As for the fractional change in length that occurs at the poles, we see that we can have an *elongation* for polytropes with n less than 3 but greater than a certain critical value. Already for $n = 2$ we have an *elongation* at the poles.

Now the volume relation for configurations with equal central densities is given by (*cf.* equation (43), I)

$$N_\omega = N_0\left(1 + \frac{9\psi_0(\xi_1)}{{\xi_1}^2}\zeta\right). \tag{106}$$

The volume relations for two configurations with equal mass is found to be

$$N_\omega(M) = N_0\left[1 + \zeta\left\{\frac{9\psi_0(\xi_1)}{{\xi_1}^2} + \frac{3(n-1)}{3-n}\left(1 - \frac{3\psi_0'(\xi_1)}{\xi_1}\right)\right\}\right]. \tag{107}$$

The coefficients of ζ occurring in (106) and (107) are tabulated in Table VI.

TABLE VI

Volume Relations

n	$\frac{N_\omega - N_0}{N_0}\cdot\zeta^{-1}$	$\frac{N_\omega(M) - N_0}{N_0}\cdot\zeta^{-1}$
0	1·5	0
1	0·9119 $\left(=\frac{9}{\pi^2}\right)$	0·9119
1·5	0·8725	1·3499
2	0·9097	1·8501
3	1·1046	...
3·5	1·2319	0·2056
4	1·3465	1·1496
5	1·5	1·5

We notice (i) *The fractional change in volume at constant* λ *for the Maclaurin and the Roche models are equal.* (ii) *For the polytropes* $n = 1$ *and* $n = 5$ *the fractional change in volume at constant* λ *equals the fractional change in volume at constant* M.

Further, by interpolation among the values in the third column of Table VI it was found that for a polytrope with $3 < n < 3{\cdot}325$ there occurs a *diminution* in volume when the configuration is set rotating. This happens

* It may be stated here explicitly that, since the change in λ is of order ζ, the expression for the oblateness ($\sigma = 0{\cdot}75\Delta_2\zeta$) is the same whether we are comparing two polytropes (with the *same* n) of equal mass or with equal central densities.

because for these polytropes the rotation is followed by such large modifications in the density distribution, in the sense of increasing the central density, that there is a consequent "shrinkage" of the whole mass.

§ 22. *The Variation of the Apparent Gravity on the Boundary of the Configuration.*—It was shown by H. von Zeipel * that *the variation of brightness* (i.e. *of the emergent flux of radiation*) *over the surface of a rotating star exactly corresponds to the variation of apparent gravity.* We shall denote by $g_{\omega,\mu}$ the apparent gravity at latitude θ when the configuration is rotating with an angular velocity ω. Then, to the first order,

$$g_{\omega,\mu} = -\left[\frac{d(V+\Omega)}{dr}\right]_{\text{Boundary}}, \tag{108}$$

where V is the gravitational potential and Ω the centrifugal potential. By equations (32), (36), I,

$$g_{\omega,\mu} = -\frac{(n+1)K\lambda^{\frac{1}{n}}}{a}\left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi_0}, \tag{109}$$

where ξ_0 is the boundary given by (38), I. We find that

$$\left(\frac{\partial\Theta}{\partial\xi}\right)_{\xi_0} = \theta_1' - \frac{v}{\theta_1'}\left\{\psi_0(\xi_1) - \frac{5}{6}\frac{\xi_1^2\psi_2(\xi_1)P_2(\mu)}{3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)}\right\}\left(\frac{d^2\theta}{d\xi^2}\right)_1 + v\left\{\psi_0'(\xi_1) - \frac{5}{6}\frac{\xi_1^2\psi_2'(\xi_1)P_2(\mu)}{3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)}\right\}. \tag{110}$$

Remembering that $\theta_1'' = 2\xi_1^{-1}\,.\,|\,\theta_1'\,|$ we obtain from (109) and (110) after some elementary transformations

$$\gamma(\mu) = \frac{g_{\omega,\mu} - g_0}{g_0} = -\zeta\left\{\frac{6\psi_0(\xi_1)}{\xi_1^2} + \frac{3\psi_0'(\xi_1)}{\xi_1} - \tfrac{1}{2}(5-\Delta_2)P_2(\mu)\right\}, \tag{111}$$

where

$$g_0 = \frac{(n+1)K\lambda^{\frac{1}{n}}}{a}\,|\,\theta_1'\,|. \tag{112}$$†

For $n=5$, (111) takes the form:

$$\gamma(\mu) = -2\zeta(1-P_2(\mu)) = -3\zeta\sin^2\theta. \tag{113}$$

---

* *M.N.*, **84**, 665, 1924.

† Using relations (101) and (106) we can rewrite our expression for $g_{\omega,\mu}$ in the form

$$g_{\omega,\mu} = \frac{GM}{a^2}[1 - \zeta\{1 - \tfrac{1}{2}(5-\Delta_2)P_2(\mu)\}], \tag{112'}$$

where a is the radius of a sphere which will have the same volume as the distorted configuration. By (92') and (99) we can express (112') differently as

$$g_{\omega,\mu} = \frac{GM}{a^2}[1 - \tfrac{2}{3}m - (\tfrac{5}{2}m - \sigma)(\tfrac{1}{3} - \cos^2\theta)], \tag{112''}$$

where the quantity m is the same as that we have already introduced in § 20. (112'') expresses merely the well-known *Clairaut's relation* for rotating fluid masses. (*Cf.* Thomson and Tait, *Natural Philosophy*, **2**, § 794, p. 362, 1903.)

The case $n=0$ has to be treated separately. From (31) and (33) we easily deduce that

$$\frac{g_{\omega,\mu}-g_0}{g_0} = -\zeta(0{\cdot}5 - 1{\cdot}25P_2(\mu)). \qquad (111')$$

From (111) and (111′) we deduce the general formula:

$$\Delta g = \frac{g_{\omega,1}-g_{\omega,-\frac{1}{2}}}{g_0} = \tfrac{3}{4}(5-\Delta_2)\,.\,\zeta. \qquad (114)$$

For $n=0$ and $n=5$ we have

$$\left.\begin{aligned} \Delta g &= 1{\cdot}875\zeta, & n&=0 \\ \Delta g &= 3\zeta, & n&=5 \end{aligned}\right\}. \qquad (114')$$

Values of Δg and the related quantities are tabulated in Table VII. In fig. 5 the variation of Δg with n is illustrated. We notice that the fractional increase in "g" at the poles passes through a maximum for some value of n between $n=0$ and $n=1$ (see fig. 6).

In connection with the variation of surface gravity over the surface, it is of interest to consider the fractional changes in surface gravity that occur when a configuration of a given mass is set rotating. If we denote by $g'_{\omega,\mu}$ the surface gravity on the boundary of the rotating configuration, then we easily verify that $g'_{\omega,\mu}$ is related to $g_{\omega,\mu}$ by means of the relation

$$g'_{\omega,\mu} = g_{\omega,\mu}\left(1+\frac{1}{2}\frac{\delta P_c}{P_c}\right). \qquad (115)$$

Numerical values for $\delta P_c/P_c$ are given in Table V. From (115) we can calculate the fractional *decrease* in g at the equator and the fractional *increase* at the poles. These quantities are tabulated in last two columns of Table VII and are illustrated in figs. 7 and 8.

TABLE VII

Variation of Apparent Gravity on the Surface

n	$\Delta g\,.\,\zeta^{-1}$	Comparison of Configurations with Equal Central Density		Comparison of Configurations with Equal Mass	
		Fractional Decrease in g at Equator $\times\zeta^{-1}$	Fractional Increase in g at Poles $\times\zeta^{-1}$	Fractional Decrease in g at Equator $\times\zeta^{-1}$	Fractional Increase in g at Poles $\times\zeta^{-1}$
0	1·875	1·125	0·75	1·625	0·25
1	2·610	1·7819	0·8281	2·4779	0·1322
1·5	2·783	2·0318	0·7512	2·8277	−0·0446
2	2·889	2·2559	0·6327	3·1964	−0·3078
3	2·978	2·6161	0·3623	...	...
3·5	2·989	2·7492	0·2398	2·1334	+0·8556
4	2·998	2·8751	0·1229	2·7657	0·2323
5	3	3	0	3	0

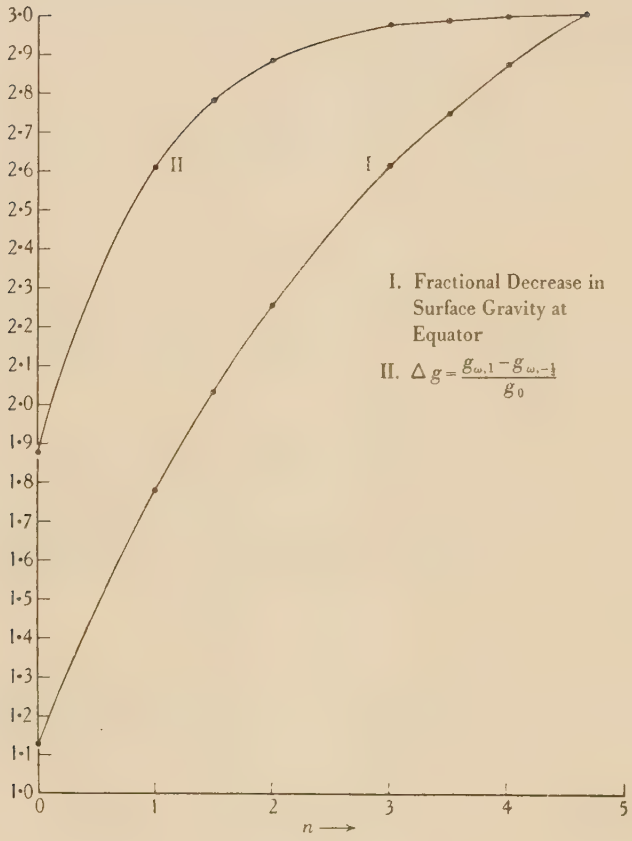

Figure 5.

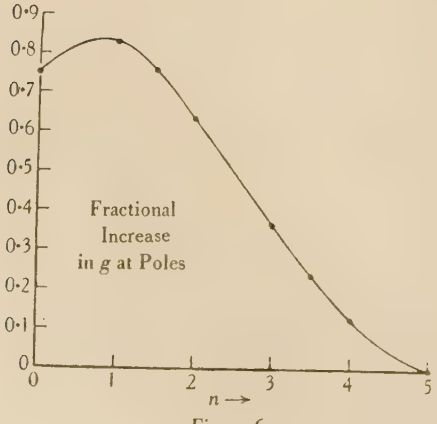

Figure 6.

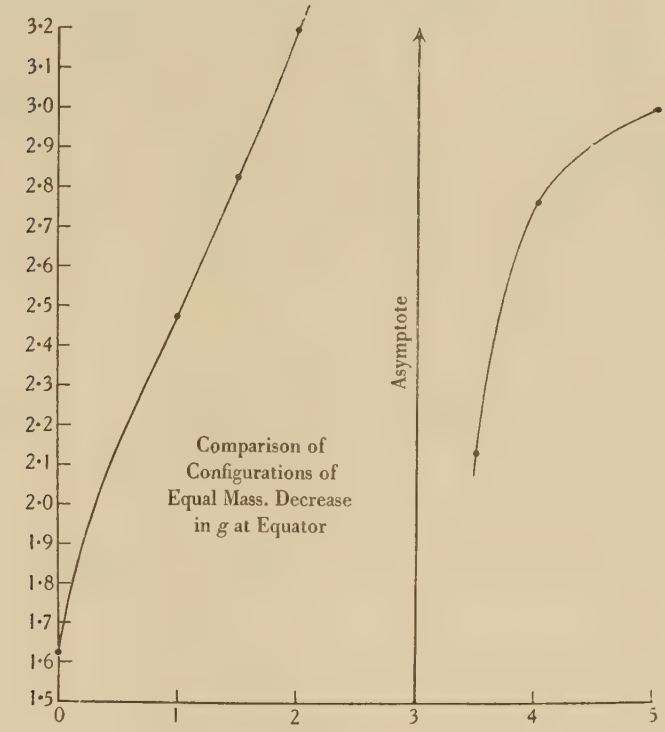

Figure 7.

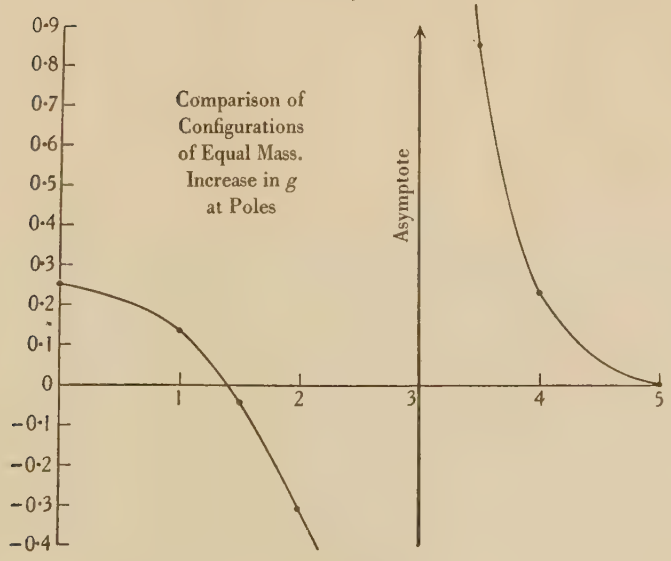

Figure 8.

Considering first the change in the surface gravity at the equator given by (115) we see that we can under certain circumstances have actually an *increase* in g at the equator. This will happen for polytropes with their index very near but greater than 3. As for the change in the apparent gravity at the poles, we see that there exist configurations which are characterised by a *diminution* in the value of the apparent gravity at the poles when it is set rotating. From fig. 8 we see that $n \sim 1{\cdot}41$ corresponds to a polytrope for which, associated with rotation, there is no change in gravity at the poles. *Polytropes with* $1{\cdot}41 < n < 3$ *are characterised by a decrease in* g *at the poles when they are set rotating.* These interesting properties of the variation of g on the boundary lends itself to much speculation, but this is not the place for them.

IV. *The Tidal Problem*

§ 23. *Distortions in External Shape as a Function of the Density Distribution.*—As a consequence of the theorem stated at the end of § 5, II, the only

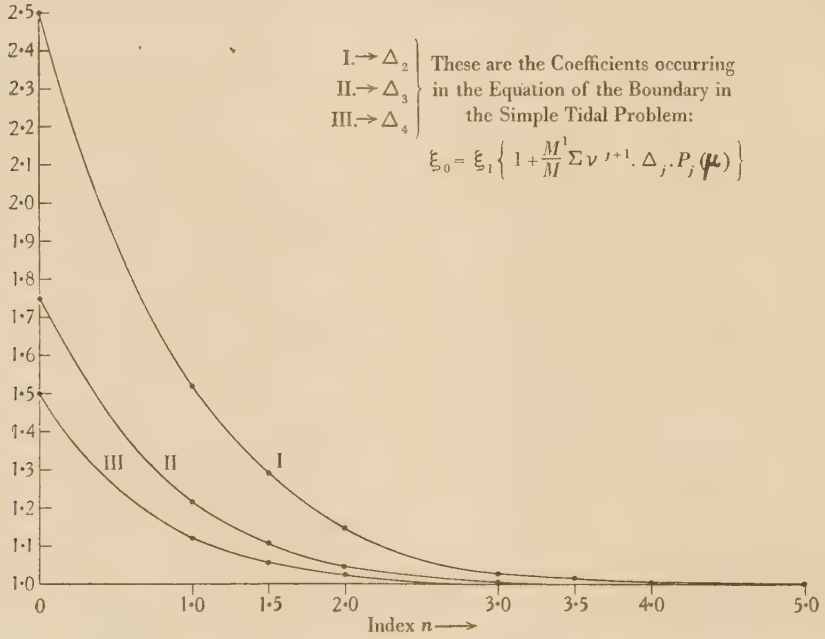

Figure 9.

feature that has to be discussed in connection with the tidal problem is the external shape.

Now the equation of the boundary can be expressed in the form:

$$\frac{\xi_0 - \xi_1}{\xi_1} = \frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot \Delta_j \cdot P_j(\mu). \qquad (116)$$

The coefficients Δ_j occurring in (116) for different values of n are collected in Table VIII, and their variation with n illustrated in fig. 9.

Table VIII

n	Δ_2	Δ_3	Δ_4
0	2·5	1·75	1·5
1	1·51985	1·2129	1·1205
1·5	1·2892	1·1079	1·0562
2	1·1482	1·0488	1·0231
3	1·0289	1·00736	1·00281
3·5	1·0147	...	...
4	1·0027	1·00047	1·00014
5	1	1	1

For the case $M' = M$ and $\nu = 0{\cdot}1$ the exact values of the distortions σ_{+1}, σ_{-1}, σ_0 at the poles and at the equator were calculated and are summarised in Table IX.

Table IX

External Shape for a Tidally Distorted Polytrope, $M' = M$, $\nu = 0{\cdot}1$

n	$\sigma_{+1} \times 10^3$	$\sigma_{-1} \times 10^3$	$\sigma_0 \times 10^3$
0	2·690	2·340	1·244
1	1·652	1·410	0·756
1·5	1·411	1·189	0·641
2	1·263	1·054	0·570
3	1·140	0·938	0·511
4	1·113	0·913	0·498
5	1·110	0·910	0·497

The results of Table IX are illustrated in fig. 10. It is rather remarkable to see how the least deviation from the uniform density distribution affects enormously the tidal distortions. This is a consequence of the Δ_i's (the coefficients of the third, fourth and fifth powers of ν occurring in the equation of the boundary) so rapidly falling down to the value unity as n deviates from zero. It would be of interest in this connection to study polytropes with $0 < n < 1$. But Emden functions for proper fractional indices are not available.

Further, when $\nu = 0{\cdot}1$ the angle χ at which the furrow develops on the external shape is about 2° (in circular measure) for $n = 0$ and about 3° for $n = 5$. The variation is not large.

§ 24. *The Variation of Apparent Gravity on the Boundary.*—In § 22 we stated a theorem of von Zeipel regarding the variation of brightness on the surface of a rotating star. We shall prove in Appendix I that an exactly

similar theorem can be formulated for a tidally distorted star. We shall therefore examine here the variation of apparent gravity on the boundary.

If we denote by $g(R, \mu)$ the apparent gravity on the tidally distorted

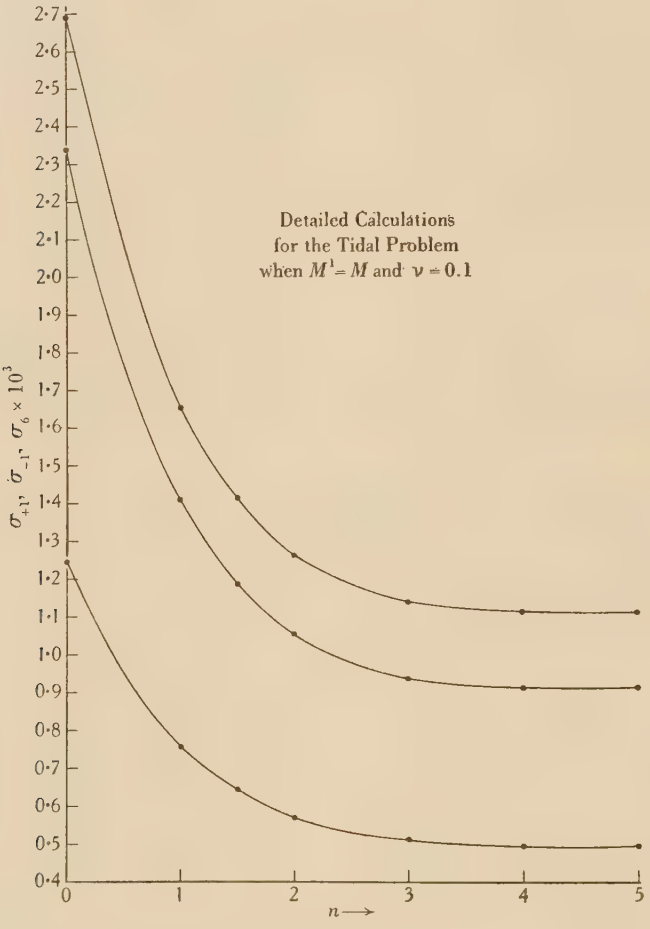

Figure 10.

configuration at latitude θ, and by g_0 the value on the undistorted configuration, then we find by an analysis similar to that in § 22 that

$$\gamma(\mu) = \frac{g(R, \mu) - g_0}{g_0} = -\frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot \{(2j+1) - (j-1)\Delta_j\} P_j(\mu). \qquad (117)$$

Two special cases of (117) may be noted

$$\left.\begin{aligned} \gamma(\mu) &= -\frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot (j+\tfrac{1}{2}) \cdot P_j(\mu), \qquad n = 0, \\ \gamma(\mu) &= -\frac{M'}{M} \sum_{j=2}^{4} \nu^{j+1} \cdot (j+2) \cdot P_j(\mu), \qquad n = 5. \end{aligned}\right\} \qquad (118)$$

Noting that the expression for $\gamma(\mu)$ is exactly of the same form as the equation of the boundary (116), we see that, similar to the formation of a furrow in the external shape (II, § 8), there is an analogous phenomenon occurring with the variation of gravity on the surface. The equation

$$\frac{\partial\gamma(\mu)}{\partial\mu}=0 \tag{119}$$

has a "non-trivial" solution. If we denote the solution of (119) by χ_g, then we clearly have (equation (48), II)

$$\chi_g=\tfrac{1}{2}\frac{7-2\Delta_3}{5-\Delta_2}\nu+O(\nu^3). \tag{120}$$

For $n=0$ $$\chi_g=0{\cdot}7\nu+O(\nu^3), \tag{121}$$

and for $n=5$ $$\chi_g=0{\cdot}625\nu+O(\nu^3). \tag{122}$$

Comparing (121) with (61) we notice the following peculiarity of the Maclaurin model :—

For the Maclaurin model the furrow in the variation of gravity is formed at angle (measured from the equator) which is exactly twice the angle at which the actual furrow in the external shape develops.

V. *The Double-star Problem*

§ 25. The double-star problem has already been discussed fully in III, but it is of interest to have all the principal distortions (III, equations (38) and (43)) numerically evaluated for one special case. As a representative example we choose $M'=2M$ and $\nu=0{\cdot}1$. The results are tabulated in Table X.

TABLE X

Distortions in a Double-star Component, $M'=2M$, $\nu=0{\cdot}1$

n	Elongation at $(\pi/2, 0)\times10^3$	Elongation at $(\pi/2, \pi)\times10^3$	Elongation at $(\pi/2, \pi/2)\times10^3$	Contraction at $(0, 0)\times10^3$
0	7·630	6·930	−0·239	3·989
1	4·673	4·187	−0·144	2·423
1·5	4·047	3·604	−0·055	1·989
2	3·707	3·288	+0·040	1·682
3	3·530	3·127	0·229	1·315
4	3·624	3·224	0·404	1·100
5	3·720	3·320	0·505	0·995

The variations of the quantities in the second, third and fifth columns of Table X with n are graphically shown in fig. 11.

Further, for this case ($M' = 2M$), $\chi_m = 2/3\chi_e$, and χ_e has already been tabulated in II (§ 8).

§ 26. *Variation of Surface Gravity.*—Without calculations we can obtain the results for the double-star problem by an application of the superposition

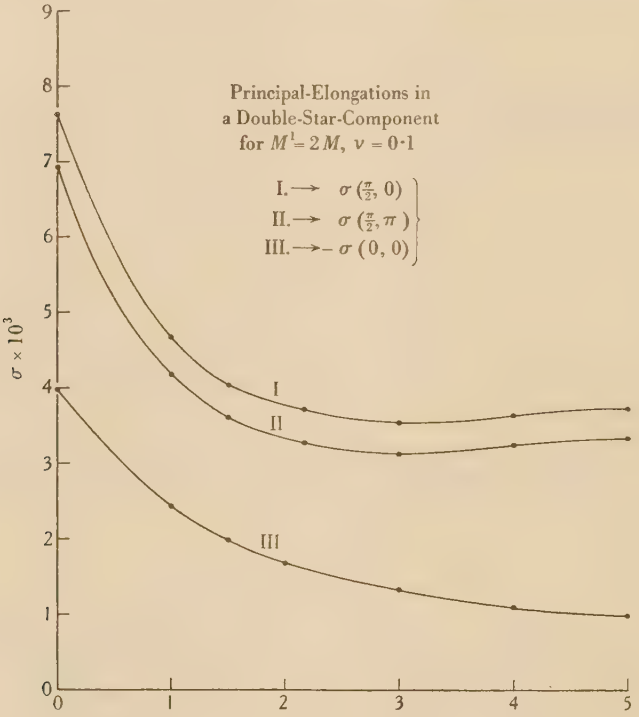

Figure 11.
(Double-Star Problem)

theorem (III, § 7). If we denote by $g(\mu, \phi)$ the apparent gravity on the boundary, then we deduce from the results of §§ 22, 24 that

$$\gamma(\mu, \phi) = \frac{g(\mu, \phi) - g_0}{g_0} = -2\left(1 + \frac{M'}{M}\right)\nu^3\left(\frac{2\psi_0(\xi_1)}{\xi_1^2} + \frac{\psi_0'(\xi_1)}{\xi_1}\right)$$
$$+ \tfrac{1}{3}\left(1 + \frac{M'}{M}\right)(5 - \Delta_2) \cdot \nu^3 P_2(\cos\theta)$$
$$- \frac{M'}{M}\sum_{j=2}^{4} \nu^{j+1}\{(2j+1) - (j-1)\Delta_j\}P_j(\sin\theta\cos\phi). \quad (123)$$

The terms in the Legendre polynomials in (123) are of exactly the same form as the function $-\sigma(\theta, \phi)$ introduced in III (§ 8). Analogous to the angles χ_m and χ_e we have now two other angles $\chi_{g,m}$ and $\chi_{g,e}$. The expressions for these latter angles can be obtained from those of the former (equations (46) and (48) in III) by writing $((2j+1) - (j-1)\Delta_j)$ wherever Δ_j occurs. Thus we have

$$\left.\begin{aligned} \chi_{g,m} &= \frac{3}{2}\frac{M'}{M+4M'}\frac{7-2\Delta_3}{5-\Delta_2}\nu + O(\nu^3), \\ \chi_{g,e} &= \frac{1}{2}\frac{7-2\Delta_3}{5-\Delta_2}\nu + O(\nu^3). \end{aligned}\right\} \tag{124}$$

Finally we notice the relation

$$\frac{\chi_{g,m}}{\chi_{g,e}} = \frac{\chi_m}{\chi_e} = 3\frac{M'}{M+4M'}. \tag{125}$$

It would be interesting to settle whether the relation (125) is a peculiarity of the polytropic model or whether it is generally valid.

Conclusion.—The underlying physical problem in these studies on "Distorted Polytropes" was to obtain some general idea regarding the nature of the dependence of the distortions in a spherically symmetrical distribution of matter consequent to a slow uniform rotation or to the presence of a tidally disturbing secondary on the original density distribution. Actually, we have solved the problem for the polytropic model mainly for this reason, that the comparatively short range for the polytropic index n {(0, 5)} includes a very large variety of density distributions including the two limiting cases. In answering the physical problem we have argued that the general nature of the dependence of the distortions on the density distribution is in effect given by their dependence on the polytropic index n (§ 18).

It is, however, quite possible that the density distributions governed by the Emden function θ have such special features that it may not be quite safe to draw general conclusions from the study of polytropes alone. In any case it was the hope that the study of the distorted polytropes have an interest of their own that made the writer undertake the rather laborious numerical work that underlies these studies.

One final remark may be permitted.

It has been found that the structure of the distorted polytropes are completely specified when four functions which we have denoted by ψ_0, ψ_2, ψ_3 and ψ_4 are known. Because of the important rôle these functions play in the theory it would perhaps be worth while to give them some special name. Personally I should suggest the name *Associated Emden Functions of the Different Orders*, with their index specified. Thus $(1-\sin\xi/\xi)$ (*cf.* equation (48), I) would on this nomenclature be called "the associated Emden function of order 0 and index 1."

APPENDIX I

A General Theorem in the Theory of Tidally Distorted Stars.—I shall prove the following theorem :—

If a mass of material in radiative equilibrium is in static equilibrium under the combined influence of its own gravitation and the tidal forces due to neighbouring masses, and if further the rate of liberation of energy $4\pi\epsilon$, *the absorption*

coefficient κ *and the gas-pressure* p *are functions of the density* ρ *and the temperature* T *only, then* ϵ *is a constant.*

For the theorem to be valid, the presence of tidally disturbing neighbouring masses is necessary. In the absence of a tidal field the theorem is not true.

The above theorem is analogous to von Zeipel's theorem in the theory of rotating stars, which states that under the same physical hypotheses as the above theorem

$$\epsilon = \text{constant} \times \left(1 - \frac{\omega^2}{2\pi G\rho}\right). \tag{1}$$

Proof.—The equations of mechanical equilibrium (II, equation (8)) can be written as

$$dP = \rho dW, \tag{2}$$

where

$$W = V + V_T. \tag{3}$$

From (2) and (3) we easily deduce (as in the proof of von Zeipel's theorem) that surfaces of constant W (*level surfaces*) are also surfaces of constant p, ρ and T. If p' is the radiation-pressure, the equation of radiative equilibrium is

$$\operatorname{div}\left(\frac{1}{\kappa\rho}\operatorname{grad} p'\right) = -\frac{4\pi\epsilon\rho}{c}. \tag{4}$$

Since P and p' are constant over a level surface we may consider p' as a function P and write (4) as

$$\sum_{x,y,z}\frac{\partial}{\partial x}\left(\frac{1}{\kappa}\frac{dp'}{dP}\cdot\frac{1}{\rho}\frac{dP}{dx}\right) = -\frac{4\pi\epsilon\rho}{c}, \tag{5}$$

which on differentiating yields

$$\frac{d}{dP}\left(\frac{1}{\kappa}\frac{dp'}{dP}\right)\cdot\frac{1}{\rho}\left(\frac{dP}{dn}\right)^2 + \frac{1}{\kappa}\frac{dp'}{dP}\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} P\right) = -\frac{4\pi\epsilon\rho}{c}, \tag{6}$$

where we have used the identity

$$\left(\frac{dP}{dn}\right)^2 = \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 + \left(\frac{\partial P}{\partial z}\right)^2. \tag{7}$$

The differential dn in (6) corresponds to the normal distance between neighbouring level surfaces. But as we have (II, equation (11))

$$\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} P\right) = -4\pi G\rho, \tag{8}$$

we can rewrite (6) in the form:

$$\frac{d}{dP}\left(\frac{1}{\kappa}\frac{dp'}{dP}\right) - \frac{1}{\rho}\left(\frac{dP}{dn}\right)^2 = -\frac{4\pi\epsilon\rho}{c} + \frac{1}{\kappa}\frac{dp'}{dP}\cdot 4\pi G\rho. \tag{9}$$

The right-hand side of (9) is a constant over a level surface. Hence the left-hand side is either constant over a level surface or is identically zero.

If it were a constant over a level surface it would follow that the differential element dn is also constant over a level surface, or, in other words, that the equipotentials are parallel surfaces. This is impossible for a tidally distorted configuration. Hence

$$\frac{d}{dP}\left(\frac{1}{\kappa}\frac{dp'}{dP}\right)=0, \tag{10}$$

or

$$\frac{1}{\kappa}\frac{dp'}{dP}=\text{constant}=\beta \text{ (say)}. \tag{11}$$

(9) now reduces to

$$\epsilon=cG\beta, \tag{12}$$

which proves the theorem.*

Q.E.D.

As a corollary to the theorem we prove :—

The variation of brightness (i.e. *of the emergent flux of radiation*) *over the surface of a tidally distorted star corresponds exactly to the variation of apparent gravity.*

We referred to this fact in § 24 before studying the variation of "g" on the surface of a tidally distorted configuration.

Proof.—Let F_n be the flux of radiant energy across a level surface at any point per unit area. Then

$$F_n=-\frac{c}{\kappa\rho}\frac{dp'}{dn}, \tag{13}$$

or by (11)

$$F_n=-\frac{c\beta}{\rho}\frac{dP}{dn}, \tag{14}$$

or again by (2)

$$F_n=-c\beta\frac{dW}{dn}.$$

Q.E.D.

APPENDIX II

Solutions of the Differential Equations :—

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)=-n\theta^{n-1}\psi_0+1$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right)=\left(\frac{6}{\xi^2}-n\theta^{n-1}\right)\psi_2$$

for n = 3·5.

* Even as we do not believe that von Zeipel's condition (1) would be fulfilled in an actual star, so also we should not expect that our new condition $\epsilon=$constant will be fulfilled in a tidally distorted star. A consequence of this would be the setting up of a *permanent* slow system of circulating currents (principally in the meridian planes), since no kind of static equilibrium is possible which is inconsistent with the condition $\epsilon=$constant. We shall not go into these delicate matters here, as it will lead us too far from our main object.

TABLE XI

Index $n=3\cdot5$

ξ	ψ_0	ψ_2	ξ	ψ_0	ψ_2
0	0	0	4·8	2·5107	5·9809
0·2	0·0066205	0·039604	5·0	2·7532	6·3595
0·4	0·025952	0·15386	5·2	3·0106	6·7562
0·6	0·05657	0·33038	5·4	3·2829	7·1713
0·8	0·09658	0·55231	5·6	3·5704	7·6050
1·0	0·14409	0·80214	5·8	3·8732	8·0572
1·2	0·19761	1·0649	6·0	4·1915	8·5293
1·4	0·25623	1·3296	6·2	4·5253	9·0202
1·6	0·31962	1·5898	6·4	4·8747	9·5304
1·8	0·38800	1·8425	6·6	5·2398	10·0601
2·0	0·46189	2·0877	6·8	5·6206	10·6093
2·2	0·54206	2·3269	7·0	6·0171	11·1779
2·4	0·62939	2·5627	7·2	6·4294	11·7661
2·6	0·77477	2·7979	7·4	6·8572	12·3739
2·8	0·82907	3·0354	7·6	7·3010	13·0011
3·0	0·94311	3·2778	7·8	7·7603	13·6478
3·2	1·06765	3·5275	8·0	8·2352	14·3139
3·4	1·2034	3·7867	8·2	8·7256	14·9993
3·6	1·3509	4·0570	8·4	9·2315	15·7039
3·8	1·5107	4·3399	8·6	9·7528	16·4277
4·0	1·6835	4·6366	8·8	10·2894	17·1705
4·2	1·8694	4·9483	9·0	10·8412	17·9323
4·4	2·0691	5·2758	9·2	11·4082	18·7126
4·6	2·2827	5·6198	9·4	11·9901	19·5121

$$\xi_1=9\cdot5358,$$
$$\psi_0(\xi_1)=12\cdot4466, \qquad \psi_2(\xi_1)=20\cdot1390,$$
$$\psi_0'(\xi_1)=2\cdot9611, \qquad \psi_2'(\xi_1)=4\cdot0706.$$

Institut for Teoretisk Fysik,
Copenhagen, and
Trinity College, Cambridge:
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THE RELATION BETWEEN THE HYDROGEN CONTENT OF THE STARS AND THEIR EQUATION OF STATE.

G. G. Cillié.

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1. *Introduction.*—There exists very little doubt at present that the high mean densities derived for white dwarfs from spectrographic and other observations are real. It has been suggested lately that the nuclei of planetary nebulæ have similar high densities.*

* See, for example, Milne, "The White Dwarf Stars," Halley Lecture, 1932 May.

The properties of stellar material under such extreme conditions were first investigated by R. H. Fowler by the application of the Fermi-Dirac statistics,* and it is now generally accepted that in white dwarfs the matter is in a degenerate and partially degenerate state. The possibility of the electron assembly becoming degenerate in ordinary stellar interiors can therefore not be neglected. It is the main object of the present paper to discuss the possible extent of degeneracy in ordinary stellar interiors—ordinary dwarfs and diffuse giants—making use of the observed and inferred dimensions of certain stars.

It is known from the investigations of Eddington † and Strömgren ‡ that in general it is possible to assign to these stars either of two values of the hydrogen content—one of the order of 30 per cent. by mass and the other approximately 100 per cent.—for which they can be constructed of perfect gas throughout. In some cases, like V Puppis and RT Lacertæ *b*, the lower figure differs appreciably from 30 per cent., and Strömgren correlates this variation with the positions of the stars in the Russell diagram.

If intermediate values of the hydrogen abundance are assigned, we shall find that the solutions of the fundamental equations are centrally condensed, showing that perfect gas conditions must break down at some point in the interior. In these cases it is of interest to determine for actual stars where the perfect gas law breaks down, especially in view of Milne's suggestion that ordinary stars have degenerate cores.§

In the first part of the paper we investigate the physical conditions under which an electron assembly is partially and completely degenerate, with special reference to stellar interior conditions. We then proceed to construct some stellar models of assumed masses, luminosities and radii and with a given hydrogen content. Following Milne, we adopt the standard model and integrate the fundamental equations from the outside with the perfect gas equation of state. The solution is continued up to the point where this breaks down. Account is taken of variation of physical conditions from model to model by using an expression for the mean temperature in an incomplete polytropic region, and calculating the mean ionisation as well as the opacity coefficient for the model at this point by means of Eddington's formulæ.

2. *Non-degenerate and Degenerate Electron Assemblies.*—According to the Fermi-Dirac statistics, the number of electrons per unit volume, n_e, at temperature T, can be written as a definite integral,‖

$$n_e = \frac{2\pi q}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{z^{1/2}dz}{e^{\alpha+z}+1}, \qquad (2.1)$$

where m is the mass of the electron, q its statistical weight, and h and k the constants of Planck and Boltzmann. This equation defines a constant α

* Fowler, *M.N.*, **87**, 114, 1926.
† Eddington, *M.N.*, **92**, 471, 1932.
‡ Strömgren, *Zs. f. Astrophys.*, **4**, 118, 1932.
§ Milne, *M.N.*, **91**, 4, 1930, and **92**, 610, 1932.
‖ Many formulæ in this section are taken from Brillouin, *Quantenstatistik*.

whose value is an indication of the extent of degeneracy of the assembly. If the energies of the particles are so small that the relativistic effect may be neglected, the pressure of the electron gas, p_e, and its energy per unit volume, U, are given by

$$p_e = 2/3\,U = \frac{4\pi q}{3h^3}(2m)^{3/2}(kT)^{5/2}\int_0^\infty \frac{z^{3/2}dz}{e^{\alpha+z}+1}. \tag{2.2}$$

When α is large and positive we may write (2.1) and (2.2) to a second order

$$n_e = \frac{2\pi q}{h^3}(2mkT)^{3/2}\Gamma(3/2)\left[e^{-\alpha} - \frac{e^{-2\alpha}}{2^{3/2}} \ldots\right] \tag{2.3}$$

and

$$p_e = \frac{4\pi q}{3h^3}(2mkT)^{3/2}kT\Gamma(5/2)\left[e^{-\alpha} - \frac{e^{-2\alpha}}{2^{5/2}} \ldots\right]. \tag{2.4}$$

Comparing (2.3) and (2.4), we find to the first and to the second order of small quantities

$$p_e = n_e kT \tag{2.5}$$

and

$$p_e = n_e kT\left[1 + \frac{e^{-\alpha}}{2^{5/2}} \ldots\right]. \tag{2.5}$$

Deviations from the classical gas law (2.5) therefore occur when $2^{-5/2}e^{-\alpha}$ becomes appreciable compared with unity. When $\alpha = 2$ the pressure differs by 2·5 per cent. from its classical value. We shall call this the point where degeneracy sets in. Hence by (2.3) degeneracy sets in where

$$\frac{n_e}{T^{3/2}} = \frac{2}{h^3}(2\pi mk)^{3/2}e^{-2}. \tag{2.6}$$

If in (2.1) and (2.2) α is large and negative, we have to a second approximation that

$$n_e = \frac{2\pi q}{h^3}(2mkT)^{3/2}\Gamma(3/2)\left[\frac{(-\alpha)^{3/2}}{\Gamma(5/2)} + \frac{\pi^2}{6(-\alpha)^{1/2}\Gamma(1/2)} \ldots\right] \tag{2.7}$$

and

$$p_e = \frac{4\pi q}{3h^3}(2mkT)^{3/2}kT\Gamma(5/2)\left[\frac{(-\alpha)^{5/2}}{\Gamma(7/2)} + \frac{\pi^2(-\alpha)^{1/2}}{6\Gamma(3/2)} \ldots\right]. \tag{2.8}$$

Comparing (2.7) and (2.8) we find

$$p_e = \left(\frac{3}{4\pi q}\right)^{2/3}\frac{h^2}{5m}\left[1 + \frac{5\pi^2}{12\alpha^2} \ldots\right]n_e{}^{5/3}. \tag{2.9'}$$

Deviations from the degenerate gas law

$$p_e = \left(\frac{3}{4\pi q}\right)^{2/3}\frac{h^2}{5m}n_e{}^{5/3} \tag{2.9}$$

occur to the extent of 2·8 per cent. or more when $\alpha > -12$. We define the system as completely degenerate when $\alpha \leqslant -12$.

3. *The Change-over from Classical to Degenerate Gas.*—From (2.3) the degree of degeneracy in an electron gas depends on the value of $n_e/T^{3/2}$, and increases with this quantity.* In ordinary stellar interiors both n_e and T increase as we proceed inwards. It is therefore probable that if degeneracy sets in at all it is a gradual process. In the present section we propose to derive the relation between the physical conditions at two points, where degeneracy is slight and where it is fairly complete, assuming that whatever the density or degree of degeneracy the electron assembly continues to behave like a gas.

Denoting the two points r_1 and r_2 by suffixes 1 and 2, then if at r_1 degeneracy is only slight and at r_2 well advanced, we find by the results of the previous section

$$\frac{n_{e_1}}{n_{e_2}} = \frac{3\pi^{1/2}}{4}\left(\frac{T_1}{T_2}\right)^{3/2} e^{-\alpha_1}(-\alpha_2)^{-3/2}. \tag{3.1}$$

Putting $\alpha_1 = 2$ and $\alpha_2 = -12$ for incipient and complete degeneracy, we find

$$\frac{n_{e_1}}{n_{e_2}} = \frac{1}{231}\left(\frac{T_1}{T_2}\right)^{3/2}. \tag{3.2}$$

In stellar interiors $T_2 > T_1$. This equation therefore shows that there is a two-hundredfold increase at least in the electron concentration between r_1 and r_2.

We find a similar relation between the electron pressures, p_{e_1} and p_{e_2}. By (2.1) and (2.2) we have that in general

$$p_e = \frac{1}{3(4\pi)^{2/3}}\frac{h^2}{m} n_e^{5/3} f(\alpha), \tag{3.3}$$

where

$$f(\alpha) = \int_0^\infty \frac{z^{3/2}dz}{e^{\alpha+z}+1} \Big/ \left[\int_0^\infty \frac{z^{1/2}dz}{e^{\alpha+z}+1}\right]^{5/3}. \tag{3.4}$$

In the two extreme cases when α is large and positive (classical case) and α large negative (degenerate case), we find respectively

$$f(\alpha_1) = 3(2\pi)^{-1/3}e^{2/3\alpha_1} \quad \text{and} \quad f(\alpha_2) = 2/5(3/2)^{5/3}. \tag{3.5}$$

Hence from (3.3)

$$\frac{p_{e_1}}{p_{e_2}} = \frac{5}{3}\left(\frac{6}{\pi}\right)^{1/3}\left(\frac{n_{e_1}}{n_{e_2}}\right)^{5/3} e^{2/3\alpha_1}, \tag{3.6}$$

or by (3.2)

$$\frac{p_{e_1}}{p_{e_2}} = \frac{15\pi^{1/2}}{8}\left(\frac{T_1}{T_2}\right)^{5/2} \frac{1}{e^{\alpha_1}(-\alpha_2)^{5/2}}. \tag{3.7}$$

Again putting $\alpha_1 = 2$ and $\alpha_2 = -12$, we find

$$\frac{p_{e_1}}{p_{e_2}} = \frac{1}{1110}\left(\frac{T_1}{T_2}\right)^{5/2}. \tag{3.8}$$

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* Degeneracy criterion, Sommerfeld, *Zs. f. Phys.*, **47**, 1, 1928; see also Milne, *M.N.*, **90**, 769, 1930.

This indicates at least a thousandfold increase in the electron pressure between r_1 and r_2. From equilibrium considerations we know that such an increase in the electron pressure inside a star must be accompanied by a similar increase in the gravitational force. This indicates that the region $r_1 - r_2$, between the gaseous and degenerate zones, is large compared with r_2.

If we adopt $a_1 = 0{\cdot}57$ and $a_2 = -6{\cdot}4$, which give deviations of the order 10 per cent. from the gas laws (1.2) and (1.3) respectively at the points r_1 and r_2, (3.7) is approximately

$$\frac{p_{e_1}}{p_{e_2}} = \frac{1}{55}\left(\frac{T_1}{T_2}\right)^{5/2}. \tag{3.9}$$

4. *The Reduction of the Fundamental Equations of Stellar Equilibrium.*— It is well known that the fundamental equations for the standard model can be reduced to an Emden polytropic equation of index 3.* In the present section we repeat the steps in that reduction, since we shall have occasion to use some of the equations in our applications later on.

The equation of radiative equilibrium may be written in the form

$$\frac{d}{dr}\left(\frac{a}{3}T^4\right) = -\frac{\kappa L_r}{4\pi c}\frac{\rho}{r^2}, \tag{4.1}$$

where κ is the mass opacity coefficient, L_r the energy generated per second inside a sphere of radius r, ρ the density and where a and c have their usual significance. The equation of mechanical equilibrium may be written

$$\frac{d}{dr}\left(\frac{R}{\mu}\rho T\right) = -GM_r\frac{\rho}{r^2} - \frac{d}{dr}\left(\frac{a}{3}T^4\right), \tag{4.2}$$

where M_r is the mass included in the sphere r, μ the molecular weight assumed constant throughout, and where R and G are the gas and gravitational constants. For the standard model we assume κ and L_r/M_r constant through the gaseous part. Then by division and using the boundary conditions $\rho = 0$ and $T = 0$, (4.1) and (4.2) give

$$\rho = \tfrac{1}{3}\frac{a\mu}{R}\frac{\beta}{1-\beta}T^3, \tag{4.3}$$

where we define

$$1 - \beta = \frac{\kappa L}{4\pi c G M}. \tag{4.4}$$

From (4.3) it is clear that $(1-\beta)/\beta$ is the ratio of radiation-pressure to gas-pressure.

By means of the relation

$$\frac{dM_r}{dr} = 4\pi\rho r^2$$

and (4.3), (4.2) may now be reduced to an Emden equation of index 3,

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta^3, \tag{4.5}$$

* See, for example, Milne, *M.N.*, **91**, 4, 1930.

where we have put

$$r = r_1 \xi, \tag{4.6}$$

$$\rho = \frac{M}{4\pi r_1^3} C^{1/2} \theta^3(\xi) \tag{4.7}$$

and

$$M_r = -MC^{1/2}\xi^2\theta'(\xi), \tag{4.8}$$

where r_1 is the total radius of the model and

$$C = \frac{48}{a\pi G^3 M^2}\left(\frac{R}{\mu}\right)^4 \frac{1-\beta}{\beta^4}. \tag{4.9}$$

The choice of r_1 in (4.6) is convenient, since that makes $\xi = 1$ the zero of θ, the value adopted by Fairclough in his tables of solutions of (4.5).*

Following Fairclough, we put

$$\omega_3 = [-\xi^2\theta']_{\xi=1}, \tag{4.10}$$

which corresponds to the outer boundary. By (4.8) and (4.9) we have, introducing numerical values,

$$\omega_3 = [\overline{35}\cdot 7512] M\mu^2 \frac{\beta^2}{(1-\beta)^{1/2}}. \tag{4.11}$$

Further

$$T = \frac{G}{4R}\frac{M\beta\mu}{\omega_3 r_1}\theta(\xi), \tag{4.12}$$

$$\rho = \rho_m \frac{\theta^3(\xi)}{3\omega_3} \tag{4.13}$$

and

$$M_r = -M\frac{\xi^2\theta'(\xi)}{\omega_3}. \tag{4.14}$$

Given the mass, luminosity and radius of a model, and provided we know the value of ω_3, we may choose the appropriate solution from Fairclough's tables, and the above formulæ give the material and temperature distributions inside the gaseous part of this model.

5. *The Mean Conditions in an Incomplete Polytropic Region.*—In the above reduction it has been assumed for mathematical convenience that κ is constant for every model. Physical theory gives κ as a definite function of the density and temperature of the material. To find the value of the opacity appropriate to any region inside a model, this expression is evaluated at some mean point inside that region. It is found convenient to choose this point as the one whose temperature has the mean value, $\bar{T}$, given by

$$\bar{T} = \int T dM_r \Big/ \int dM_r, \tag{5.1}$$

* Fairclough, *M.N.*, **91**, 55, 1930, and **93**, 40, 1933. These tables have been a great help in the numerical applications.

the integration extending through the region in question. The mean ionisation potential, ψ_1, for the region will be calculated for this point as well.

By (4.12) and (4.14) we are therefore led to consider

$$\bar{\theta}_{\xi_1}^{\xi_2} = \frac{\int_{\xi_1}^{\xi_2} \theta \frac{d}{d\xi}(\xi^2\theta')d\xi}{\int_{\xi_1}^{\xi_2} \frac{d}{d\xi}(\xi^2\theta')d\xi}, \tag{5.2}$$

where ξ_1 and ξ_2 are the limits of the region. Integrating successively by parts and using (4.5), it can be shown that *

$$-\int_{\xi_1}^{\xi_2} \theta \frac{d}{d\xi}[\xi^2\theta']d\xi = \left[\xi^3\theta^4 + 2\xi^2\theta\theta' + 2\xi^3\theta'^2\right]_{\xi_1}^{\xi_2}. \tag{5.3}$$

Hence by (5.2)

$$\bar{\theta}_{\xi_1}^{\xi_2} = \frac{\left[\xi^3\theta^4 + 2\xi^2\theta\theta' + 2\xi^3\theta'^2\right]_{\xi_1}^{\xi_2}}{\left[-\xi^2\theta'\right]_{\xi_1}^{\xi_2}}. \tag{5.4}$$

If we take as upper limit $\xi_2 = 1$ where $\theta(1) = 0$ and $[-\xi^2\theta']_{\xi=1} = \omega_3$, and as the other limit $\xi_1 = \xi$, we find

$$\bar{\theta} = \frac{2\omega_3^2 - 2\xi^2\theta'\theta\left[1 + \xi\frac{\theta'}{\theta} + \frac{1}{2}\frac{\xi\theta^3}{\theta'}\right]}{\omega_3 + \xi^2\theta'}. \tag{5.5}$$

In general when $\theta(\xi_0) = 0$ we have to replace $2\omega_3^2$ by $2\omega_3^2/\xi_0$ in (5.5). In this case if θ is the regular solution of (4.5) for which $\theta'(0) = 0$ and $\theta(0)$ is finite and positive, (5.5) reduces to Eddington's formula for the mean temperature.

If now we know ω_3 and ξ or $\theta(\xi)$, we are in a position to calculate $\bar{\theta}$ by means of the appropriate solution from Fairclough's tables. From this we may calculate the mean temperature $\bar{T}$ and the corresponding density $\bar{\rho}$ by using (4.12) and (4.13).

6. *The Setting in of Degeneracy in the Standard Model.*—The relation between the electron concentration and the temperature for incipient degeneracy is expressed by (2.6). In converting this into a relation between ρ and T we notice that for stellar interior conditions, where all atoms are practically completely ionised, there are two limiting cases.

(*a*) When the constituent atoms are of high atomic weight, the gas-pressure, p_g, approximately satisfies

$$p_g = p_e = n_e kT. \tag{6.1}$$

(*b*) If hydrogen only is present

$$p_g = 2p_e = 2n_e kT. \tag{6.2}$$

* Milne, *Handbuch d. Astrophys.*, 3 (1), p. 194, footnote.

If μ is the mean molecular weight, we find in both cases

$$p_g = \frac{R}{\mu}\rho T.$$

Hence in all cases

$$\frac{R}{\mu k}\rho > n_e > \frac{R}{2\mu k}\rho. \tag{6.3}$$

We shall adopt the upper limit and keep in mind a possible correction in the case of an excess of hydrogen. Substituting in (2.6), we find the condition for incipient degeneracy

$$\frac{\rho}{T^{3/2}} = \frac{\mu k}{R}\frac{2(2\pi mk)^{3/2}}{h^3}e^{-2}. \tag{6.4}$$

Eliminating ρ between this relation and (4.3), which holds for the standard model, we find the temperature where degeneracy sets in

$$T_{\text{deg}} = 1{\cdot}08 \times 10^9\left(\frac{1-\beta}{\beta}\right)^{2/3} \tag{6.5}$$

on putting in numerical values.

It is well known that $(1-\beta)/\beta$ is the ratio of radiation-pressure to gas-pressure in the standard model. If radiation-pressure is unimportant, as is the case for models containing a large proportion of hydrogen, (6.5) shows that $T_{\text{deg}} < 10^9$. Also, we know that, in general, radiation-pressure is relatively more important for models of high luminosity than for those of low luminosity. The temperature at the point where degeneracy sets in in a giant is therefore higher than the temperature at the corresponding point in a dwarf.

Stars of high luminosity and late spectral type (giants) have radii large compared with those of stars of low luminosity and corresponding type (dwarfs). On account of the homologous property of the fundamental equations, the temperature inside a model with a large radius is generally lower than that inside a model with a small radius, except perhaps for a small region round the centre. The above result therefore leads us to expect that degeneracy affects relatively larger regions in dwarfs than in giants.

We note that if T_{deg} and ω_3 for a model are known, (4.12) gives the value of θ_{deg}. $\bar{T}$ and $\bar{\rho}$ may then be determined as explained at the end of section 5.

7. *Applications to Actual Stellar Models.*—In the present section we give the results of calculations similar to those of Eddington.* We construct the gaseous part of centrally condensed models corresponding to actual stars as regards M, L and r_1, assuming that they contain first 50 per cent. and then 80 per cent. by mass of hydrogen, the rest of the material being a mixture of elements whose mean is iron. Eddington's procedure was to take ω_3 as known and equal to $\omega_3{}^0$ and to predict L. Our problem is the converse, namely, to find ω_3 or the appropriate member of the family of solutions of (4.5) in each case from the given M, L and r_1.

* Eddington, *loc. cit.*

According to Eddington the dominant ionisation potential, $-\psi_1$, is given by

$$\frac{\psi_1}{k\bar{T}} = \log_e\left\{\frac{\sigma_0}{2}\left(\frac{h^2}{2\pi mk\bar{T}}\right)^{3/2}\right\}, \tag{7.1}$$

where σ_0 is the number of electrons per c.c. for the density $\bar{\rho}$, and may be calculated for any hydrogen-iron mixture when the degree of ionisation is known. From the value of $-\psi_1/k\bar{T}$ the so-called "guillotine factor," g, may be read off from Eddington's table,* and also the range of the spectrum on which most of the weight of the opacity depends. This in turn gives the range of energy of the important capture levels, or the effective charge, $Z\epsilon$, of the iron ions which contribute most of the opacity.

If AH is the total mass per iron atom, the opacity coefficient at the "mean point," neglecting the "guillotine" correction, is given by

$$\kappa' = [25{\cdot}2191]\frac{Z^2}{A}\frac{\bar{\rho}}{\mu\bar{T}^{7/2}(1+f)}, \tag{7.2}$$

where f is the ratio of the total number of ions to the total number of electrons. Finally, applying the guillotine correction, (4.4) gives

$$1-\beta = \frac{\kappa' L}{4\pi cGMg}. \tag{7.3}$$

The value of $1-\beta$ given by (7.3) should agree with the one originally adopted.

In practice the problem is, given the M, L, r_1 and hydrogen content of a model, to find the value of $1-\beta$, ω_3, T_{deg}, etc., taking account of physical conditions in the interior especially as expressed by (7.1) and (7.2), the values of $\bar{T}$ and $\bar{\rho}$ being obtained as shown in section 6. Solutions are found by trial and error and are only approximate. They serve to show, however, the general tendency for the giant stars to have smaller regions affected by degeneracy than the dwarfs, confirming the conclusion in section 6.

Nature of Models with 50 per cent. and 80 per cent. Hydrogen

Star	$\log_{10} L$	$\log_{10} M$	Radius (r_1)	%H	$1-\beta$	ω_3	T_{deg}	ξ_{deg}	$M_{r\,\text{deg}}/M$
Kreuger 60 *b*	31·4875	32·7292	$2{\cdot}149\times10^{10}$	50	0·0012	0·63	$1{\cdot}22\times10^{7}$	0·25	0·9
Sun	33·5775	33·2978	$6{\cdot}951\times10^{10}$	50	·0034	1·31	$2{\cdot}45\times10^{7}$	·10	·4
				80	·0024	0·79	$1{\cdot}94\times10^{7}$	·12	·7
Algol	35·9958	33·9313	$2{\cdot}18\times10^{11}$	50	·035	1·58	$1{\cdot}18\times10^{8}$	·005	·08
				80	·018	1·25	$7{\cdot}51\times10^{7}$	·015	·17
Capella	35·5775	33·9190	$8{\cdot}47\times10^{11}$	50	·034	1·60	$1{\cdot}18\times10^{8}$	·001	·08
				80	·017	1·22	$7{\cdot}22\times10^{7}$	·003	·11

In the first four columns of the table are given the data for the stars taken from Eddington's paper. Column 5 contains the percentage hydrogen

* Eddington, *M.N.*, **92**, 364, 1932.

assumed. In column 6 we give that value of $1-\beta$ which is found to give the best solution of the above equations. In the case of Kreuger 60 *b* the solution is only very approximate owing to lack of solutions of (4.5) for which $\omega_3 < 1$. The column illustrates the increase in the relative importance of radiation-pressure with increasing luminosity.

The eighth column gives the temperature at the point where degeneracy sets in, and shows that for a given hydrogen content this increases from dwarf to giant.

In columns 9 and 10 are given the approximate values of the distance of the point, where degeneracy sets in, from the centre in terms of the total radius and the fraction of the total mass in each case affected by degeneracy.

The increase in the hydrogen abundance decreases the radiation-pressure, and so makes the solution more centrally condensed and lowers T_{deg}.

Eddington and Strömgren have shown that in general there exist two values of the hydrogen content—one approximately 100 per cent.—for which the star can be constructed by the use of the regular solution of (4.5) (Emden-Eddington models). Except in the case of low-luminosity dwarfs like Kreuger 60 *b*, these models are gaseous throughout, the physical conditions never satisfying the degeneracy criterion.

By the above calculations we know that the solution in the gaseous part is centrally condensed if the hydrogen abundance is intermediate between these two values. In this case the perfect gas equation of state breaks down at some point.

It is further interesting to note that, in the cases of the Sun, Algol and Capella, Strömgren's determinations of the hydrogen content are minimum values. Solutions with less hydrogen in the gaseous part are collapsed, and there is then no possibility of making the calculated and observed radii agree.

For red dwarfs like Kreuger 60 *b*, Eddington and Strömgren find that, adopting the regular solution of the fundamental equations with the perfect gas equation of state, the material becomes degenerate at some point in the interior. It should be pointed out that any solution of the fundamental equations, in which it is assumed that the gas laws hold to the centre and for which it appears *a posteriori* that these laws are violated before the centre is reached, cannot be taken as a fair representation of the gaseous region itself. It can be shown that in general there exists a minimum value for the hydrogen content of these red dwarfs as well. These correspond to collapsed solutions in the gaseous part.

8. *The Degenerate Core.*—The fundamental equations of stellar equilibrium become very intractable in the region where the matter is partially degenerate, and no attempt is here made to construct a complete model. The results of section 3 show that the density and the pressure increase considerably between the points where degeneracy sets in and where it is complete. It is therefore possible that for some models (dwarfs) degeneracy is incomplete even at the centre.

From the investigations of Kothari * we know that when stellar material becomes degenerate, the mass absorption coefficient is more than that given

* Kothari, *M.N.*, **93**, 61, 1932.

by the non-degenerate opacity calculated for the same physical conditions, but the increased conduction of heat by the electrons actually tends to decrease the "effective" opacity coefficient. If the matter in a star becomes degenerate this would decrease the temperature gradient in comparison with the case when degeneracy is absent. Degenerate cores will therefore be comparatively more uniform in temperature than ideal highly compressible ones composed of perfect gas.

This result combined with those of the previous section brings out one important difference between gaseous models and those with degenerate cores. For the former it is found that the temperatures and densities inside giants are much lower than those inside dwarfs.* For the latter the opposite is the case. The giants have round their centres a small region of high density and temperature ($T > 10^8$). The dwarfs have larger cores which are more diffuse and cooler. These models, with a small degenerate core, are therefore more in accordance with a theory of energy generation in which the generating process is favoured by extreme physical conditions.

Several criticisms have been made of Milne's theory of centrally condensed stars. The most important of these depends upon the reduction of the fundamental equations to Emden polytropic equations and on certain mathematical theorems in connection with their solutions.† This objection becomes important in view of the fact that with the excess of hydrogen required by physical theory, radiation-pressure is relatively unimportant in all stars, and β is therefore approximately unity in all cases.‡ The discussion of the setting in of degeneracy in stars in section 3 has indicated the great extent of the transition region. In this region, however, the fundamental equations cannot be reduced to an Emden polytropic equation unless degeneracy is well advanced. The above objection is therefore not valid as it stands.

9. *Summary and Conclusion.*—Beginning with given values of M, L and r_1 for certain stars, we attempt to find the extent to which the interiors of such stars are affected by degeneracy. This makes it necessary to investigate the conditions under which an electron assembly becomes degenerate in stellar interiors. It is found that at least a thousandfold increase in the electron-pressure is required between the point where degeneracy becomes appreciable and where it is practically complete. This suggests that for an actual star, whose interior is not perfectly gaseous throughout, there is an extensive region where the material is in a state between that of a perfect gas and a state of complete degeneracy.

Eddington and Strömgren have shown that for ordinary stars—diffuse giants and ordinary dwarfs, excluding those of smaller mass—there exist two values of the hydrogen abundance with which they can be constructed, of perfect gas throughout. We find that with intermediate values of the hydrogen content the solutions of the fundamental equations are centrally

* See Eddington, *I.C.S.*, p. 297, 1926.

† Eddington, *Zs. f. Astrophys.*, 3, 129, 1931. Also Strömgren, *M.N.*, 91, 466, 1931 ; Cowling, *ibid.*, 91, 472, 1931 ; Russell, *ibid.*, 91, 739, 1931.

‡ See Milne, *M.N.*, 91, 479, 1931.

condensed in the gaseous part, showing that the gas laws break down at some point in the interior. Any star therefore either contains as much hydrogen as is predicted by Eddington and Strömgren and is gaseous throughout, or it has some intermediate hydrogen abundance and has a degenerate core. In general there exists a minimum value for the hydrogen content for every star.

For the non-gaseous models our results show that for diffuse giants very small regions—of the order 0·005 r_1—are affected by degeneracy. In dwarfs the region is relatively larger—0·4 r_1. These figures are to some extent dependent on the assumed hydrogen content.

For these centrally condensed solutions we find, on using Kothari's results, that diffuse giants have relatively small but hot and dense cores. Dwarfs, on the other hand, have large cores which are relatively cool and diffuse. It is pointed out that this has a bearing on the difficulty encountered in connection with the process of energy generation inside gaseous models.

In conclusion, it is a pleasure to record my thanks to Professor E. A. Milne for helpful discussion and criticism.

Hertford College, Oxford :
1933 *June* 2.

THE RELATIVE INTENSITIES OF INTERSTELLAR CALCIUM AND SODIUM LINES.

C. S. Beals.

The nature and properties of interstellar matter are becoming increasingly well understood as a result of extensive observations of the so-called "stationary" or "detached" lines of sodium and calcium which are a common feature of the spectra of early-type stars. The hypothesis of a tenuous cloud of matter, uniformly distributed throughout the galaxy and made up of atoms similar in nature, and probably in relative abundance, to those found in the Earth's crust and the atmospheres of stars was made familiar by the theoretical work of Eddington,* and has recently been placed on a very sound observational basis by Plaskett and Pearce. † The only serious discrepancy which has so far appeared is to be found in the comparatively large intensities of interstellar sodium lines, which on the basis of Eddington's calculations as well as those of Gerasimovič and Struve ‡ were expected to be extremely weak relative to those of calcium.

Up to the present, observations of interstellar lines have had to do mainly with determinations of radial velocity and with eye estimates of relative

* *Internal Constitution of the Stars*, p. 383, 1926.
† *Pub. D.A.O.*, 5, No. 3, 1933.
‡ *Ap. J.*, 69, 7, 1929.

intensity, though it has been recognised that spectrophotometric measures would be eventually required in order to make the data complete. For the purpose of supplying some of the necessary quantitative determinations of intensity, the writer has recently undertaken an extended programme of observation of stars in which interstellar lines appear. So far the observations have been largely confined to the ordinary region of observation and have had to do with the calcium lines in relatively faint stars, but a number of brighter objects have been observed for the purpose of determining the relative intensities of interstellar calcium and sodium lines. In the present paper results are given of measures of the lines of sodium and calcium in the two stars ξ Persei and 9 Camelopardalis.

The spectrograms in the ordinary region of observation were taken with a single prism spectrograph having a camera of 42 cm. focal length and a dispersion at λ 3933 of 33·5 A. per mm. In the visible region the spectrograph was used in its three-prism form with a camera of 71 cm. focal length and a dispersion of 30·7 A. per mm. at λ 5892. This particular choice of instruments was made in order to keep the dispersion as closely as possible the same in the two regions. A slit width of from 0·025 to 0·050 mm. was used, and the lines as they appeared on the plate were extremely fine ; so that the line contour, whose form was determined almost entirely by the properties of the spectrograph, had an extreme width of from 0·06 to 0·10 mm. The plates used in the ordinary region were mainly Eastman 33, while Ilford Special Rapid Panchromatic plates, sensitised with ammonia, were used in the visible region.

The method of calibrating the plates and the microphotometer used for making the measurements have been described in a previous publication.* The microphotometer slit was used at a width of 0·06 mm., and since the magnification was approximately 17 this corresponded to a distance on the plate of 0·0035 mm. The necessarily finite resolving power of the microphotometer combined with the narrowness of the lines presumably causes systematic errors of unknown amount in the absolute values of the intensities, but the effects of such errors in comparing intensities of sodium and calcium lines should be largely minimised by the use of approximately equal dispersions in the two regions.

The results of the measurements are expressed as "total absorptions" in terms of the integral $\int_{v\,\min}^{v\,\max} (1-r)dv$, where r represents the ratio of the intensity of a point in the contour to that of the neighbouring continuous spectrum, and v represents displacement from the centre of the line expressed as a Doppler shift in km./sec. No particular significance attaches to this choice of units, since the mechanism of absorption producing the true contours of interstellar lines is not as yet completely understood. The transformation to wave-length or frequency units is, of course, readily performed.

The numerical values of total intensity, which were determined graphically from the contours by means of a planimeter, are shown in Table I.

* *M.N.*, **92**, 196, 1932.

Values for λ 3968 of calcium have not been included, since this line is blended with Hε, and higher dispersion spectrograms will probably be required in order to separate the two contours with sufficient accuracy. Each line has been measured on five plates, and the results for each plate as well as the means are included in the table. Individual deviations from the mean are rather large, as might be expected from the smallness of the area measured on each plate, but the results as a whole are reasonably consistent and leave little doubt that the intensities of the sodium lines are equal to or slightly stronger than those of calcium.

TABLE I

Intensities of Interstellar Lines

ξ Persei (*H.D.* 24,912) *spectral type* O7

Plate No.	λ 3933 *Ca* II Intensity	Plate No.	λ 5889 *Na* Intensity	λ 5895 *Na* Intensity
18967	17·7	21162	18·7	15·6
18969	20·2	21163	22·9	16·2
19330	12·2	21164	14·8	14·4
19420	14·1	21165	16·2	16·9
20959	16·4	21166	13·9	11·8
Mean	16·1	Means	17·3	15·0

9 *Camelopardalis* (*H.D.* 30,614) *spectral type* O9

Plate No.	λ 3933 *Ca* II Intensity	Plate No.	λ 5889 *Na* Intensity	λ 5895 *Na* Intensity
19436	29·5	21084	32·6	32·6
19437	37·2	21085	36·7	30·0
19466	29·7	21127	25·7	22·4
20177	22·2	21128	27·1	23·7
21226	25·1	21174	30·2	25·1
Mean	28·7	Means	30·5	26·8

This result scarcely comes as a surprise as it has been recognised for some time that the sodium lines were of at least the same general order of intensity as those of calcium. The theoretical interpretation of the relative intensities of the two sets of interstellar lines involves three factors: (1) The relative abundance of sodium and calcium atoms in the interstellar matter; (2) the relative numbers of sodium and calcium atoms in neutral and ionised states; and (3) the relative values of the absorption coefficients for *Na* and *Ca* II. Both the discussion of Eddington and that of Gerasimovič and

Struve suggest that the amount of latitude involved in the calculations, due to uncertainty in the numerical values of these three factors, may be sufficient to account for the apparent discrepancy in the observed and calculated intensities of interstellar lines. It is hoped that the quantitative measures given above may be of some help in defining the problem more clearly, while additional physical data on the properties of the atoms of sodium and calcium should eventually make a satisfactory solution possible.

It is rather surprising to find that, within the probable errors of the determination, the intensity ratio $\frac{5889}{5895} = 1{\cdot}15$ is approximately the same for the two stars in which the absolute values of the intensities are so different. The interpretation of the variation of this ratio with absolute intensity depends to a considerable extent on the mechanism producing the true line contour ; and while a first attack on the problem has been made by Unsöld, Struve and Elvey,* the nature of the mechanism is still incompletely understood. The writer hopes to extend the measures of this ratio to include a considerably larger range of absolute intensities.

Dominion Astrophysical Observatory,
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THE TEMPERATURES OF THE NUCLEI OF PLANETARY NEBULÆ.

R. H. Stoy, B.A., Isaac Newton Student, University of Cambridge.

(Communicated by Sir Arthur Eddington)

Introduction

The nuclei of the planetary nebulæ have such high surface temperatures that the ordinary methods of determining stellar temperatures cannot be applied to them. Zanstra † has developed two methods by which the nuclear temperature can be found from comparision of the nebular and nuclear spectra on a calibrated slitless spectrogram. In this paper a method of determining the nuclear temperature from observations of the nebular spectrum alone is developed. This method rests on the same theoretical assumptions as Zanstra's, but appears to be simpler and of wider application. This formula when applied to existing observational data yields temperatures rather lower than those at present current. Some of the assumptions on which the various formulæ are based are briefly examined, and it is found that some of the very high temperatures that have been derived for some of the planetary nuclei are not justifiable.

* *Zs. f. Astrophysik*, **1**, 314, 1930.

† H. Zanstra, *Zeit. für Astrophysik*, **2**, 1, 1931 ; or *Pub. D.A.O. Victoria*, **4**, 209, 1931.

§ 1. *Derivation of the Formula from Zanstra's Model*

Zanstra's model of a planetary nebula consists of a hot central star radiating like a black body surrounded by a very extensive tenuous atmosphere composed of hydrogen with which is mingled a very small proportion of other ions, mainly those of doubly ionised oxygen. As a high degree of ionisation is observed in the actual nebulæ, it is supposed that the greater part of this hydrogen is dissociated into protons and electrons, whilst on account of the very low density in the nebula, the high dilution of the radiation flowing through it and the very short life of an excited hydrogen atom, the overwhelming majority of the remaining neutral hydrogen atoms are supposed to be in their normal state.

Three main processes are supposed to go on continuously in this nebula. The first of these processes is the recombination of free electrons and protons, resulting in the emission of the Balmer series and the formation of neutral hydrogen atoms ; the second is the ionisation of neutral hydrogen atoms from their ground state by ultra-violet quanta (by which, in this paper, is meant quanta whose wave-length is less than 912 A., and which are therefore capable of ionising an hydrogen atom from its ground state) ; and the third process consists of inelastic collisions between free electrons and the foreign ions (*i.e.* the non-hydrogen ions) which result in the excitation of the ions, which, owing to the tenuity of the nebular material and the consequent rarity of collisions, relapse to their normal state by the emission of radiation.

The nebula is supposed to have reached a steady state in which the first process makes as many hydrogen atoms as the second destroys, whilst the kinetic energy of the photo-electrons freed by the second process is absorbed by the third. This perhaps requires some explanation. The velocity distribution of the free electrons will tend to the Maxwellian. Expelled photo-electrons get entangled in this crowd of free electrons and are quickly robbed of their initial energy, whilst electrons that have suffered a non-elastic collision and lost energy will quickly regain it at the expense of their fellows. In any interval of time as much energy is supposed fed into this electron gas by the hydrogen photo-electrons as is taken out of it by the inelastic collisions that are the direct cause of the emission of the so-called nebulium lines.

The ultra-violet quanta which cause the ionisation of the hydrogen can be divided into two classes: those which were emitted by the star (either directly or after transformation by one of the foreign ions) and those which have been emitted in the nebula as a result of captures of free electrons on the lowest level by protons.

Zanstra makes three subsidiary assumptions.

The first is the assumption that there is sufficient nebular matter present to ensure that every quantum emitted, whose wave-length is less than 912 A., is absorbed and causes either directly or indirectly the ionisation of one and only one normal hydrogen atom. With this assumption, the rate of ionisation by ultra-violet quanta emitted in the nebula will equal the rate at which free electrons are captured on the first level by protons. Hence, as in a

steady state the rates of recombination and of ionisation must be equal, we can assert that the rate at which protons capture free electrons on the second and higher levels is equal to the rate at which ultra-violet quanta are emitted by the star.

The second assumption is that each such recombination of a free electron with a proton, in which capture occurs on the second or higher level, results in the emission of one and only one Balmer quantum. This assumption taken with the former one enables us to say that in any second the number of ultra-violet quanta emitted by the star is equal to the number of Balmer quanta emitted by the nebula. This translated into mathematics gives equation (4).

The number of quanta with frequencies between ν and $\nu + d\nu$ emitted by the star in one second is

$$4\pi R^2 \cdot \tfrac{1}{4} c \rho_\nu d\nu \cdot \frac{1}{h\nu}, \tag{1}$$

where R denotes the radius of the star, T its temperature, and ρ_ν the energy density of full radiation corresponding to a temperature T. By Planck's law

$$\rho_\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/kT} - 1}. \tag{2}$$

Hence the total number of ultra-violet quanta emitted by the star in one second is

$$4\pi R^2 \int_{\nu_0}^{\infty} \tfrac{1}{4} c \cdot \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/kT} - 1} \cdot \frac{1}{h\nu} \cdot d\nu, \tag{3}$$

where ν_0 denotes the frequency of light which is just able to ionise a hydrogen atom.

The number of quanta emitted per second by the nebula in the light of one of the Balmer lines is $\frac{L_P}{h\nu_P}$, where L_P denotes the total energy emitted per second by the nebula in the light of that line and ν_P denotes the frequency of that light. Consequently the total number of Balmer quanta emitted per second by the nebula is

$$\sum_{\text{Hyd}} \frac{L_P}{h\nu_P},$$

where the summation includes all the lines of the Balmer series and also a term $\int_{\nu'}^{\infty} \frac{L_P'}{h\nu} d\nu$ for the continuous spectrum at the head of the Balmer series, L_P' denoting the energy in light of frequencies between ν and $\nu + d\nu$ that the nebula emits per second in this spectrum. Thus Zanstra's first two assumptions give us the equation

$$4\pi R^2 \int_{\nu_0}^{\infty} \tfrac{1}{4} c \cdot \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/kT} - 1} \cdot \frac{1}{h\nu} \cdot d\nu = \sum_{\text{Hyd}} \frac{L_P}{h\nu_P}. \tag{4}$$

The third assumption that Zanstra makes is that all the kinetic energy of the freed hydrogen photo-electrons is used in stimulating the nebulium spectrum. This gives us equation (7), obtained as follows :—

The kinetic energy of an electron freed by a quantum of frequency ν is given by

$$\tfrac{1}{2}mv^2 = h\nu - h\nu_0, \tag{5}$$

and the number of such quanta emitted by the star in one second is given by (1), so that the total energy made available in one second for the stimulation of the nebulium spectrum is

$$4\pi R^2 \int_{\nu_0}^{\infty} \tfrac{1}{4}c \cdot \frac{8\pi h}{c^3} \cdot \frac{\nu^3}{e^{h\nu/kT} - 1} \cdot \frac{1}{h\nu}(h\nu - h\nu_0)d\nu. \tag{6}$$

But the energy radiated per second by the nebula in the nebulium spectrum is simply $\sum_{\text{Neb}} L_P$, where L_P is defined as before, and the summation extends over all the non-hydrogen lines. Equating these two expressions for the energy emitted per second we get

$$4\pi R^2 \int_{\nu_0}^{\infty} \tfrac{1}{4}c \cdot \frac{8\pi h}{c^3} \cdot \frac{\nu^2(\nu - \nu_0)}{e^{h\nu/kT} - 1} d\nu = \sum_{\text{Neb}} L_P. \tag{7}$$

Dividing equation (7) by equation (4) we get

$$\frac{\displaystyle\int_{\nu_0}^{\infty} \frac{\nu^3 - \nu_0\nu^2}{e^{h\nu/kT} - 1} d\nu}{\displaystyle\int_{\nu_0}^{\infty} \frac{\nu^2}{e^{h\nu/kT} - 1} d\nu} = \frac{\displaystyle\sum_{\text{Neb}} L_P}{\displaystyle\sum_{\text{Hyd}} \frac{L_P}{\nu_P}}. \tag{8}$$

If we put

$$x = \frac{h\nu}{kT} \tag{9}$$

equation (8) may be written

$$\frac{\mathcal{J}_2 - x_0\mathcal{J}_1}{x'\mathcal{J}_1} = \frac{\displaystyle\sum_{\text{Neb}} L_P}{\displaystyle\sum_{\text{Hyd}} L_P\left(\frac{\nu'}{\nu_P}\right)}, \tag{10}$$

where ν' denotes some chosen frequency, say that of $H\beta$, and

$$x_0 = \frac{h\nu_0}{kT}, \qquad x' = \frac{h\nu'}{kT},$$

while

$$\mathcal{J}_1 = \int_{x_0}^{\infty} \frac{x^2}{e^x - 1} dx, \qquad \mathcal{J}_2 = \int_{x_0}^{\infty} \frac{x^3}{e^x - 1} dx. \tag{11}$$

As in equation (10) only the ratios of the various L_P are involved, we may replace them by the observed relative integrated intensities I_P of the various

images found from a slitless spectrogram, or, if there is no very marked variation in the sizes of the various monochromatic images, by the ordinary relative intensities measured with a slit spectroscope. We thus obtain the equation

$$\frac{\mathcal{J}_2 - x_0\mathcal{J}_1}{x'\mathcal{J}_1} = \frac{\sum_{\text{Neb}} I_P}{\sum_{\text{Hyd}} I_P \frac{\nu'}{\nu_P}}, \qquad (12)$$

from which, given the observed relative intensities I_P of the various nebular lines, we may deduce the temperature of the central star.

The right-hand side of this equation is independent of T and can be determined directly from observation, whilst the left-hand side, which contains no observed quantities, is a function of T only, whose value may be tabulated for various values of T. From this table, or from a graph plotted from it, the temperature corresponding to any observed value of the right-hand side can be found immediately. Approximately values of the left-hand side of equation (12) for a series of values of T are given in Table I.

TABLE I

Values of $\mathcal{J} \equiv \frac{\mathcal{J}_2 - x_0\mathcal{J}_1}{x'\mathcal{J}_1}$ *for* x *corresponding to* 912 *A.*, x' *to* 4861 *A.* (H_β)

$T = 15{,}000°$	$\mathcal{J} = 0{\cdot}59$	$T = 40{,}000°$	$\mathcal{J} = 2{\cdot}0$	$T = 65{,}000°$	$\mathcal{J} = 3{\cdot}6$
20,000°	0·85	45,000°	2·3	70,000°	4·0
25,000°	1·1	50,000°	2·6	80,000°	4·8
30,000°	1·4	55,000°	2·9	90,000°	5·5
35,000°	1·7	60,000°	3·2	100,000°	6·2

§ 2. *Application of the Formula*

Unfortunately there is at present very little observational data to which formula (12) may be applied. It may be of interest, however, to apply it to the line intensities found by Plaskett * for N.G.C. 7027, a very bright planetary with no well-defined nucleus to which Zanstra's methods cannot be applied (Berman's † application of Zanstra's formulæ is probably fallacious; finding no trace of a nuclear spectrum he used instead the faint continuous spectrum emitted by the nebulosity which may be due to free-free electron switches as predicted theoretically by Cillié ‡). Using Plaskett's values for the ordinary lines and Berman's † for the ultra-violet lines which Plaskett has not measured, formula (12) gives a temperature of about 43,500° for the stimulating star of N.G.C. 7027. If we use Plaskett's values of the line intensities for the Orion Nebula in formula (12) we get a kind of mean stimulation temperature for that nebula of 17,000°. (This estimate should be

* H. H. Plaskett, *Publ. D.A.O. Victoria*, **4**, 187, 1928.
† L. Berman, *Lick Obs. Bull.*, **15**, 86, 1930.
‡ G. G. Cillié, *M.N.*, **92**, 820, 1932.

slightly increased, as it has been impossible to include the ultra-violet lines, which Plaskett has not measured.)

§ 3. *Comparison with Zanstra's Methods*

Zanstra's methods of determining the nuclear temperature depend on the determination from a slitless spectrogram of the ratio of the energy emitted in a monochromatic image (L_P) to the energy per unit frequency emitted in the stellar spectrum $\left(\frac{\partial L_s}{\partial \nu}\right)$, where $\left(\frac{\partial L_s}{\partial \nu}\right)$ is measured at the same frequency as L_P. His methods have the great advantage that comparisons are only made between lights of the same frequency that have traversed the same optical paths ; no subsidiary standard black body is needed. On the other hand, it is difficult to compare accurately light in the very narrow nuclear spectrum with that in the nebular image, especially when there is a very great difference in intensity between them. For very few of the bright planetaries will the nuclear spectrum be at all comparable in intensity with the nebular images, especially near N_1, N_2 and H_α, which make the most important contributions to the quantities from which the nuclear temperature is determined.

Formula (12), however, is independent of the nuclear spectrum, and can be applied equally well to the many small bright nebulæ in which the nucleus is inconspicuous as to those in which the nuclear spectrum is strong enough to be measured. It has the disadvantage of requiring a subsidiary standard black body, but when once that drawback has been overcome, the reduction of the observations is much simpler and quicker than those needed by Zanstra's methods.

§ 4. *Consideration of the Subsidiary Assumptions on which the Formula is Based*

We have assumed that each stellar quantum whose frequency is greater than ν_0 is absorbed and causes the ionisation of one and only one hydrogen atom. This cannot be true unless there is a great abundance of normal hydrogen present in the nebula. Moreover, some of the ultra-violet radiation is used in keeping the oxygen and other atoms present ionised. This radiation, however, is not completely lost to the hydrogen, for if the nebula is in a steady state, for each ionisation of such an atom there is a corresponding recombination which results in the emission of a quantum of radiation of frequency lower than that of the stellar quantum absorbed, yet still of high enough frequency to ionise a hydrogen atom. [This is true of all the ions involved in the actual nebulæ with the exception of N^+ and N^{++} which make only a partial return.] The photo-electric absorption coefficient decreases with increasing frequency, so that the foreign ions actually help the hydrogen to get its full share of the very high frequency stellar quanta. It is to be noted that if *He* and He^+ serve the hydrogen in this way (and presumably they do), the helium lines should be considered as belonging to the nebulium

spectrum when we apply the formula, for they have drawn the energy that they emit from the same source, viz. the energy remaining after the hydrogen atoms have been ionised.

The second assumption was that each recombination of free electron and proton in which capture occurs on the second or higher level results in the emission of one, and only one, Balmer quantum. The justification of this assumption given below also requires a great abundance of normal hydrogen, for it requires that the free path of any Lyman quantum emitted be small compared with the dimensions of the nebula.

If an electron is captured on the second level a quantum in the Balmer continuous spectrum is emitted and the newly formed atom relapses to the ground state with the emission of an L_α quantum (L_α denoting the first line in the Lyman series). Owing to the scarcity of hydrogen atoms in the right state to absorb it, the Balmer quantum will presumably find its way straight out of the nebula, but the L_α quantum will be absorbed by a normal hydrogen atom. It will, however, quickly be re-emitted, for owing to the conditions prevailing in the nebula, the absorbing atom is unlikely to be disturbed during its short excited life of about 10^{-8} seconds. After many such scatterings, the L_α quantum will eventually find its way out of the nebula.

If an electron is captured on the third level, a quantum in the continuous Paschen spectrum is emitted and the newly formed atom may reach the ground state either by emitting H_α followed by L_α or by emitting L_β. The H_α and L_α quanta would find their way out of the nebula unchanged, but before the L_β quantum could escape it would be absorbed by a normal hydrogen atom, which it would raise to the third level. In about eight cases out of nine this atom would re-emit L_β, but in the ninth it would emit H_α followed by L_α. If the free path of a Lyman quantum is small compared with the dimensions of the nebula, the probability that any L_β quantum emitted is split into H_α and L_α before it leaves the nebula is very large, and the chance of its survival as L_β correspondingly small. Thus after thirty free paths, the chance of survival of an L_β quantum is less than 2 per cent., so that no great error will be made in supposing that all the L_β quanta emitted get split into H_α and L_α quanta.

The last of the series of transitions made by an electron captured on one of the other levels must be one to the ground state, as a result of which a quantum in the Lyman series is emitted. If this quantum is not L_α—if it were the electron must have come from the second level and so already have caused the emission of a Balmer quantum—it will be reabsorbed, and by the process described for L_β will be split into a number of quanta, one of which is L_α and another a Balmer quantum.

It appears, then, that the assumptions, from which equation (4) is derived, are likely to hold if, and only if, there is a great abundance of hydrogen in the nebula.

The third assumption on which equation (12) is based is that all the kinetic energy of the freed hydrogen photo-electrons is used in stimulating the nebulium spectrum. This cannot be absolutely true, as the electrons captured by the protons on the second and higher levels have some kinetic

energy which will be converted into radiation in the Balmer and other continuous hydrogen spectra; but as the chance of capture varies like $1/v^4$, only slow electrons are likely to be caught, so that the total energy of the electrons dissipated in this way is likely to be small. It is possible to make some correction for this by including in the nebulium spectrum the energy emitted in the continuous Balmer and Paschen spectra, while, at the same time, including in the hydrogen spectrum the number of quanta emitted in the Balmer continuous spectrum.

§ 5. *Criticisms of some of the Applications of Zanstra's Formulæ*

Equation (4) is essentially the equation from which Zanstra calculates the temperature of the central star from observations of a recombination spectrum. He applies it separately to the observed spectra of *H*, *He* and He^+. We have seen that the application of equation (4) to the hydrogen spectrum is probably justifiable if there is a large abundance of hydrogen in the nebula, but the above considerations do not justify its application to the spectrum of ionised helium. We cannot assert that each suitable stellar quantum ionises an He^+ ion, since the He^+ and O^{++} ions have to share such quanta, nor can we find the number of quanta emitted in the Balmer series of ionised helium; we can only guess it. We cannot observe the Balmer series which is in the ultra-violet, nor can we observe the whole of the Paschen series of ionised helium, and even if we could, it would only tell us the number of quanta emitted in the first line of the Balmer series and not the total relevant number of recombinations that occur in one second.

The realisation that Zanstra's recombination formula cannot be applied satisfactorily to the observed spectrum of He^+ (of which 4686 is the most important line) is of some importance, since it is from its application to 4686 that Zanstra * and Berman † get temperatures of the order of 70,000° to 80,000° for some of the planetary nuclei, and Beals ‡ gets equally high temperatures for Wolf-Rayet stars and novæ. The breakdowns in the assumptions upon which the formula depends, which have already been pointed out, offer us no escape from these high temperatures; on the contrary, they rather suggest that the application of the formula would lead to temperatures that were too low.

It appears probable, however, that there might be a breakdown in the basic assumption that the *He* II spectrum is emitted as the result of the recombination of free electrons and alpha particles, *the balancing ionisations being accomplished by stellar quanta*. The stellar quanta may not be the only ionising agency at work.

Of the ten planetaries in which Campbell and Moore § have observed the splitting and broadening of the spectral lines, nine of them show the line 4686, while the tenth (N.G.C. 6543) has 4686 concentrated in the nucleus as

* H. Zanstra, *loc. cit.*
† L. Berman, *loc. cit.*
‡ C. S. Beals, *M.N.*, **92**, 677, 1932.
§ Campbell and Moore, *Publ. Lick Obs.*, **13**, 178, 1918.

a strong emission band. The only other nebula which shows 4686 and which has been tested with high dispersion is N.G.C. 2440, which, to quote Campbell and Moore, "is faint and not well situated for the required exposure." (To the list given by Campbell and Moore on p. 178, N.G.C. 1535 and N.G.C. 6572 might be added.)

The most natural explanation of these broadened and split spectral lines is that they are due to Doppler effect, the nebular matter moving outwards from the star with velocities of the order of 30–100 km./sec. A similar explanation has been given of the broadening of the bands in Wolf-Rayet spectra, only here the outward velocities required are much greater, while for novæ there is very strong evidence of the outward motion of matter with large velocities.

For an ionised helium atom to ionise itself by collision with another atom, it need have a velocity of about 50 km./sec. only, so that it appears possible that the helium in these expanding shells becomes ionised mainly by collision. Such ionising collisions may be supposed as resulting rather from the encounter of molecules of one cloud of gas with those of another cloud, through which the first cloud is passing, than from the random velocities of molecules within one cloud. In this case the results obtained by applying Zanstra's formula would be entirely misleading. While some of the hydrogen atoms would also be ionised by collision, it is likely that this number would be less in proportion to those ionised by stellar quanta than for helium, so that the application of Zanstra's formula to the hydrogen spectrum would be less misleading, especially as for any given temperature, the temperature given by this formula is less sensitive to change in the observed quantity when applied to hydrogen than it is when applied to ionised helium.

Equation (7) is essentially the equation from which Zanstra calculates the temperature of the central star from observations of the nebulium spectrum, and the one which forms the base of his method for deriving an approximate nuclear temperature from the difference in magnitude between the central star and the nebula, a method by which he has derived temperatures as high as 140,000° for some of the planetary nuclei. To obtain his formula, Zanstra assumes that the photographic brightness of the nebula bears a constant ratio to the intensity of N_1 and N_2. Equation (12), however, suggests that this ratio will not be constant, but will vary almost linearly with T. Moreover, Zanstra assumes that the photographic magnitude of the central star is proportional to the spectral energy at a definite wave-length, "the isophotic wave-length," which he assumes to be the same for all the nuclei. As B. Vorontsov-Velyaminov * has pointed out, this is most unlikely to hold true for nuclei which display strong emission bands superposed on a faint continuous spectrum. Further, the application of a formula of this nature assumes that the planetary nebulæ form a homogeneous class, an assumption whose truth is by no means self-evident. No great weight, then, can be attached to temperature estimates obtained from this formula, even if they are based on homogeneous and reliable observational data, which Zanstra's estimates certainly were not.

* B. Vorontsov-Velyaminov, *Russian Astronomical Journal*, **8**, 122, 1931.

§ 6. *Conclusion*

Table II contains all the direct estimates of nuclear temperatures that have yet been given. The temperatures given in columns 2 to 6 are due to Berman,* with the exception of those marked with a (Z) which are due to Zanstra,† and those enclosed in brackets which are due to V. Ambarzumian,‡ but which are based on Berman's observations. The temperatures in columns 7 and 8 were obtained by applying formula (12) to the line intensities observed by Plaskett § or Berman,* together with certain assumptions about the unobserved lines. Unfortunately none of these estimates can be regarded as anything more than intelligent guesses based on the application of a theoretical formula to insufficient observational data.

TABLE II

Estimates of the Nuclear Temperature

Nebula	(Recombination)			Nebulium	Colour Temperature	Formula (12)	
	H	*He* I	*He* II			$H_\alpha : H_\beta = 4{\cdot}0 : 1$	$H_\alpha : H_\beta = 5{\cdot}8 : 1$
6572	43,500 41,000 (Z)	43,500 34–41,000 (Z)	...	41,500 38,000 (Z)	...	34,000	27,000
6543	36,100 37,500 (Z)	36,400	...	32,000 35,000 (Z)	30,000	24,000	18,500
II 4593	25,000	...	...	24,000	25,000	20,000	15,500
6826	26,500	29,500	...	27,000	25,000	27,000	22,000
7009	40,000	39,000	70,000 (115,000)	40,000	40,000(?)	51,000	41,000
7662	43,000	...	81,000	51,000	...	35,000	29,000
7027	> 52,000	> 40,000	> 86,000 (165,000)	> 53,000	...	...	43,500

It has already been shown that the application of the recombination method to *He* II is unjustifiable and perhaps misleading; the application to *He* I rests on no better theoretical grounds. The intensity of H_α, the biggest contributor to the number of quanta emitted in the Balmer series, was not measured either by Zanstra or by Berman. In order to obtain a value for the nuclear temperature, Berman assumed an intensity of H_α found by extrapolation of the observed-intensity curve of the Balmer series from H_β to H_η. This makes the ratio of the intensity of $H\alpha$ to that of H_β about 2 : 1, which is probably too low. The only measurements of this ratio are

* L. Berman, *loc. cit.*
† H. Zanstra, *loc. cit.*
‡ V. Ambarzumian, *Pulkovo Obs. Circ.*, No. 4, 1932.
§ H. H. Plaskett, *loc. cit.*

due to Plaskett, who found a value of 4·0 : 1 for the Orion Nebula and 5·8 : 1 for N.G.C. 7027. It seems reasonable to expect that the values for other planetaries will lie between these limits. Similarly when applying the nebulium method, Berman, having no measurements of lines in the red, had to ignore them, though it would appear from Plaskett's measurements for N.G.C. 7027 that 6548 and 6584 (the *N* II equivalents of the *O* III N_1 and N_2 lines) radiate quite an appreciable amount of energy. Fortunately, however, the temperatures derived from Zanstra's methods are very insensitive to errors in the observed quantities for temperatures above 30,000° [the quantity to be observed for the hydrogen method varies approximately as T^3, and for the nebulium method approximately as T^4], so that we may expect the temperatures shown in columns 2 and 5 to be of the right order, with the possible exception of those for N.G.C. 7027 (see § 2).

The temperatures given in columns 7 and 8 are open to similar objections due to lack of observational data. These temperatures are not so reliable, as for this method the quantity to be observed varies approximately linearly with T. Column 7 gives the value of the temperature obtained from the observed-line intensities when a value of 4·0 : 1 is adopted for the ratio of the intensity of H_α to that of H_β, while column 8 gives the temperature found by adopting a value of 5·8 : 1 for this ratio. The temperatures given in column 7 are probably more accurate than those given in column 8 for the lower temperatures, and *vice versa* for the higher ones. Where the intensity of N_1 has not been measured, it has been taken as three times that of N_2, which is the theoretical value according to Stevenson.* It has been necessary to ignore all other unmeasured lines and also the Balmer continuous spectrum, which for some of the nebulæ might make an appreciable contribution to the number of quanta emitted in the hydrogen spectrum.

Formula (12) appears to give values for the temperature which are rather smaller than those derived by Zanstra's methods. The observational data is, however, far too scanty for us to attempt to draw any conclusions from this, such as the breakdown of the initial assumptions through lack of abundance of hydrogen, or through our ignoring a possible accumulation of hydrogen atoms in the 2S metastable state, etc. When more extensive observational data becomes available, it may be possible to draw some such conclusion from differences in the temperatures obtained by these three alternative methods which, though they rest on the same theoretical assumptions, are observationally independent.

In conclusion I should like to express my deep-felt thanks to Sir Arthur Eddington, not only for his interest and advice in connection with this paper, but also for his constant encouragement and for the many stimulating discussions on planetary nebulæ that I have been privileged to share.

The Observatory, Cambridge :
1933 *June* 1.

* A. F. Stevenson, *Proc. Roy. Soc.*, A, **137**, 298, 1932.

A RAPID METHOD FOR DETERMINING VISUAL BINARY ORBITS.

Henry Norris Russell.

The problem of determining the orbit of a binary star, visual, spectroscopic or eclipsing, differs radically from that of finding the orbit of a planet or comet. In the latter case the percentage accuracy of the observations is great, and elaborate theoretical developments make it possible to obtain a solution from observations extending over but a small arc of the orbit, and of the minimum number which suffice analytically for a determinate solution.

In the former, the errors of observation, though small in absolute amount, are large in proportion to the quantities sought. No reliable solution can be made until the observations cover a large part of a revolution (for a visual binary) or several complete revolutions (for spectroscopic or eclipsing systems), and a number of observations far in excess of the theoretical minimum is always requisite. The problem is not to represent three or four observations exactly, but to fit a score or more as well as possible. Graphical methods are therefore of great usefulness, though, of course, they must always be based on a firm analytical foundation.

A satisfactory method for handling such a problem should meet the following conditions :—

1. It should be based on the whole series of available observations, not on a selected few.

2. The time is always far more accurately known than the other data—position angle, distance, radial velocity or stellar magnitude. It should therefore be kept explicitly in view in the discussion. This is always done in plotting the velocity or light-curves of spectroscopic or eclipsing binaries, and the advantages of a similar treatment of visual binaries are great.

3. Curves drawn to represent the observations may play an important part in the solution. An experienced computer is likely to draw these curves nearly of a theoretically possible shape, but, even so, the object of the solution is to fit the observations rather than an empirical curve.

4. Known relations should be taken into account in drawing these curves ; for example, a constant period is assumed in assembling observations of a spectroscopic or eclipsing binary. For a visual binary the law of areas, $s^2\frac{dp}{dt} = \text{constant}$, similarly serves to connect the plots of angle and distance against the time. Good judgment regarding the relative weighting of observations and the rejection of bad ones is here essential.

5. The analytical solution should be made, so far as possible, for the parameters which define the *curve*. As soon as a first approximation has been obtained the resulting curve should be computed and compared with

the observed points. Further approximations should then be made until the representation is satisfactory.

6. The passage from the curve-parameters to the conventional elements of the system should be deferred, if possible, until a good representation has been secured. The computation of the ephemeris should precede that of the elements.

7. When a curve is drawn empirically to represent a set of observed points, the ordinate of the curve is usually fixed with much greater relative precision than its slope—that is, the function itself is better determined than its derivative, and *a fortiori* than its second derivative. Formulæ involving second derivatives of observed quantities are therefore likely to give poor results. On the same principle, the height of a maximum is usually better determined than its time (assuming, of course, that there are adequate observations in the vicinity). For curves of a simple form, and above all for straight lines, these objections almost disappear. Analytical methods which transform the observed curve into one of simpler shape are therefore advantageous. Sometimes the new curve may be obtained by shifting a curve of fixed shape, or one of a one-parameter set of such curves vertically and horizontally. In this case curves on transparent paper are useful. Here, again, the observed points, rather than the free-hand curve, should be transformed into the new system. A little intelligence is required to determine what discordances from the transformed points correspond to permissible residuals from the old ones.

8. Least-square solutions should never be attempted until the resources of graphical adjustment have been fully exploited. It is a fair criterion of a good least-square solution that all the corrections should be of the same order of magnitude as their probable errors, and certainly a criterion for a satisfactory preliminary solution that this should be the case. The computer thus escapes the heart-breaking experience of finding, after the heavy labour of least-squares, that the second-order terms were sensible, and the whole job must be repeated.

9. Last, but not least, the work should not be the mere mechanical application of standard formulæ. Intelligent appreciation of the accuracy available in the data and required in the discussion is continuously required. To get the best results out of data of low precision is as much of an art as a science.

For a visual binary, the writer has found great advantages in returning to Herschel's old device of interpolating curves—plotting the distance s and position angle p separately against the time. Free-hand curves are drawn to represent the plotted points. Reading s and p for equal intervals of time, the successive areas described, which are sensibly equal to $s_1 s_2 (p_2 - p_1)$, are computed with a slide-rule (which, by the way, is an invaluable aid in all such work). The adoption of a mean value for the constant of areas, and the adjustment of the two curves to represent the observations as well as may be, while satisfying the law of areas, are mainly matters of judgment based on the character of the observations—particularly the older ones—and no formal rules can be given for them.

When the final interpolating curves have been adopted, s and p may be read for convenient epochs and rectangular co-ordinates $x = s\cos(p - p_0)$, $y = s\sin(p - p_0)$ calculated with the slide-rule and plotted. The angle p_0 should be taken 90° from the position angle of the line adjoining the first and last plotted points.

The resulting points should lie on an ellipse. The general equation of an ellipse may be written

$$y = a + bx \pm c\sqrt{(x_0 - x)(x - x_1)}. \tag{1}$$

If y_1 and y_2 are the values of y corresponding to a given x, we have

$$\tfrac{1}{2}(y_1 + y_2) = a + bx, \tag{2}$$

$$\frac{(y_1 - y_2)^2}{x_0 - x} = 4c^2(x - x_1). \tag{3}$$

From the free-hand curve drawn through the plotted points, values of y_1 and y_2 may be read for various x's. The mid-points of the chords parallel to $x = 0$ should lie on the straight line (2) and a and b can thus be found. The intersection of (2) with the curve gives x_0 (or sometimes x_1). Plotting $\dfrac{(y_1 - y_2)^2}{x - x_0}$ against x another straight line is obtained which gives x_1 and c^2.

All the parameters defining the apparent ellipse (1) are now known, and it may be plotted by points. In the writer's experience, this ellipse almost always agrees very closely with the empirically adjusted curve.

The centre of this ellipse is given by

$$X = \tfrac{1}{2}(x_0 + x_1), \qquad Y = a + \tfrac{1}{2}bX. \tag{4}$$

For any point on it, we have

$$x = X + l\cos\phi, \qquad y = Y + bl\cos\phi + cl\sin\phi, \tag{5}$$

where $l = \tfrac{1}{2}(x_1 - x_0)$ and ϕ is an auxiliary angle. From the geometry of the ellipse, it follows that ϕ differs only by a constant from the eccentric anomaly E in the true orbit; that is,

$$\phi = E + \phi_1. \tag{6}$$

At the periastron point, $ex = (e - 1)X$, $ey = (e - 1)Y$, $\phi = \phi_1$. Hence

$$\begin{aligned} el\cos\phi_1 &= -X, \\ el\sin\phi_1 &= -(Y + bX)/c, \end{aligned} \tag{7}$$

which determine e and ϕ_1.

The area of the ellipse is $A = \pi cl^2$. If $s^2\dfrac{dp}{dt} = h$, the period P is $2A/h$. If $h°$ corresponds to the measurement of p in degrees instead of radians,

$$P = 360cl^2/h° \tag{8}$$

and the mean motion is

$$n = h°/cl^2. \tag{9}$$

The ephemeris may now be directly computed from (5), (6) and

$$n(t-T)=E-e\sin E, \tag{10}$$

where T is the time of periastron passage.

It is most convenient to work both ways from assumed values of E, say 15° apart, avoiding the solution of Kepler's equation. s and p may then be plotted against $t-T$, and T determined so as to give the best fit with the observed points directly. The representation of the observations is, in the writer's experience, usually satisfactory. If not, changes may be made in the constants of the apparent ellipse. Insight into the geometrical conditions, aided by the method of "false position," is a better guide to the next trial than any formal rule.

When the representation of the observations appears to be as good as can be obtained, the elements may be determined as follows: Let α_1 be the length of the semi-diameter of the apparent orbit which passes through periastron, and χ_1 its position angle, and let α_2, χ_2 be those of the conjugate semi-diameter. Then

$$\begin{aligned}
\alpha_1\cos(\chi_1-p_0)&=l\cos\phi_1,\\
\alpha_1\sin(\chi_1-p_0)&=bl\cos\phi_1+cl\sin\phi_1,\\
\alpha_2\cos(\chi_2-p_0)&=-l\sin\phi_1,\\
\alpha_2\sin(\chi_2-p_0)&=-bl\sin\phi_1+cl\cos\phi_1.
\end{aligned}$$

(The numerical values usually have been already computed in preparing the ephemeris.)

The usual elements may then be obtained by Zwiers's method, by the equations

$$\begin{aligned}
&\beta=\alpha_2(1-e^2)^{-\frac{1}{2}},\\
&a^2+b^2\pm 2ab=\alpha_1{}^2+\beta^2\pm 2\alpha_1\beta\sin(\chi_2-\chi_1),\\
&\cos\gamma=b/a,\\
&\sin^2\lambda=(a^2-\alpha_1{}^2)/(a^2-b^2),\\
&\tan(\chi_1-\Omega)=\tan\lambda\cos\gamma,\\
&\tan(\chi_2-\Omega)=-\cot\lambda\cos\gamma.
\end{aligned}$$

A very brief account * of this method was published by the writer in 1917, but seems to have attracted no notice, though Luyten † has recently applied somewhat similar reasoning. The writer's conviction that it is really useful, especially for slow moving pairs, may justify this more formal presentation. Experience has shown it to be rapid, two complete orbits having once been computed in one day.

Princeton University Observatory:
1933 *May* 16.

* *A.J.*, **30**, 131, 1917.
† *A.J.*, **42**, 1, 1933.

THE MOTION OF THE MOON, 1923–1931.

Ernest W. Brown.

1. The chief object of this paper is to present a summary of results obtained from the discussion of 3673 occultations of the Moon observed during the nine years from the beginning of 1923, when the new tables of the Moon's motion were first used in the almanacs, to the end of 1931. The normal equations for each lunation are given and combined in three different ways to form annual means. The results chosen as definitive for the occultations are compared with the corresponding values of the longitude and latitude deduced from the Greenwich and Washington meridian observations, and annual mean values are deduced from a combination of all the material. The corresponding changes in the Earth's rate of rotation on the assumption that these annual differences are due to such changes are shown. Certain annual oscillations in the longitude and fourteen-month oscillations in the latitude are exhibited and discussed.

2. Strictly speaking, the accidental error of an occultation is small. The observer simply records the time of its occurrence so that there are no instrumental errors directly involved. But there are errors due to the inequalities of the limb and to faulty positions of the stars occulted. The systematic effects of these errors can undoubtedly be almost entirely eliminated and the remaining effects can be then treated as accidental. It has not been found practicable to make such corrections as yet; in fact, one object in view here is an attempt to find out whether a new discussion with all known corrections applied would materially alter the results. The errors have been treated as accidental here, but a knowledge of their general character has been used in making combinations so as to avoid, as far as possible, their systematic effects.

3. It was hoped that a full analysis of the differences between the results from occultations and from the meridian observations would give a clue to the source of the annual term which has become so marked a feature of the results from the occultations in the last few years. While a small seasonal effect can be traced in the meridian observations, its phase is sufficiently different from that of the occultations to make a common source very doubtful. And a newly discovered effect in the latitudes has complicated the problem. There is little doubt that this latter oscillation has a period substantially longer than a year, probably fourteen months. The meridian observations show a similar effect, but of smaller amount and having the opposite phase.

The hypothesis that these might be tabular errors is scarcely probable, since it would demand that they were largely cancelled by the observing conditions in the meridian observations. It is true that Loomis and Stetson

have found some considerable differences between time determinations at Washington and Greenwich depending on the hour angle of the Moon, which may possibly be evidence of a world-wide effect, but it is difficult to see how they could cause the differences between the occultations and the meridian observations. Another difficulty is the considerable magnitudes of the latter ; they appear to be too large to be attributed to such sources as naturally occur to the investigator. And if it be assumed that they have periods of a month or shorter, converted into longer periods by the distribution and method of discussion, as is possible, the actual coefficients would have to be still larger. For some time past I have felt that in these differences something of considerable interest was to be found, and therefore that it was worth while to give the data in considerable detail. A systematic difference between the two sets of observations of another kind was pointed out earlier (*Yale Obs. Trans.*, **3**, 227), but it is stated by Spencer Jones that with his new reduction of the Newcomb collection of occultations this effect is considerably smaller.

4. *Material.*—For the years 1923–26, the results of reductions of occultations contained in a paper to appear soon in the *Astronomical Journal* are used. This paper is a revision, with additions, of a similar summary for these years which appeared in *A.J.*, 902. The revision consists mainly in a fresh calculation of the residuals when a uniform amount, 7″·00, is added to the tabular mean longitude. Certain errors, which did not sensibly affect the annual mean values, are also corrected.

The additions are as follows : To 1924, for which only 56 occultations were previously available, the results from 136 collected and reduced by the Bond Club, and published in *A.J.*, 987. To 1925, which previously contained 108 occultations not well distributed over the year, the results from 66 additional reductions (which include some duplications of previous reductions) made by the computing section of the B.A.A. and published in vol. xliii, p. 100, of the *Journal* of the Society. To the years 1924–26 have been added the results of the Cape occultations (those for 1923 had been included by Innes), published in *M.N.*, **92**, 846, together with a few others which were found in the literature.

For these five years the normal equations for each lunation have been formed on the plan adopted by Dr. Brouwer and myself for the years 1928–31, the results of which are given without any change from those appearing in *A.J.*, 948, 955, 970, 991 ; these publications, however, did not contain the coefficients of the normal equations for each lunation. The actual numbers of observations utilised are shown by the sums of the second and fourth columns of Table I for the separate lunations and the corresponding sums in Table II for the annual values. As in the later years, results giving large residuals and bright limb occultations are rejected. An exception to the latter, however, occurs in 1923–24 when certain bright limb observations of α Tauri and α Leo, both first magnitude stars, are included.

The Greenwich meridian results are taken from the volume for 1927 of the Greenwich observations for the years 1923–27, and from the annual volumes for 1928–30 ; the annual mean for 1931 is taken from the report

of the Astronomer Royal for that year. The limb observations are kept separate from those of Mösting A.

The Washington meridian observations are given in *A.J.*, 939 for the years 1923–29, and in *A.J.*, 957, 974 for the two following years. For both Greenwich and Washington the published means by lunations are used here. To the Washington annual means for the years 1923–24, the quantity 0″·83 has been added to $\delta\lambda$ and 0″·10 to $-\delta\beta$, in order to take care of a difference between the results from the 6-inch and 9-inch transit instruments (see *A.J.*, 868).

5. In all cases the residuals have been obtained on the same plan, namely, the addition to the tabular mean longitude of an amount which will reduce the average residual for the year to less than 1″. An addition of 7″·00 has been used in the occultations for the years 1923–28 and of 6″·00 for the years 1929–31. By this method the periodic changes in the elliptic and other inequalities due to a change in the mean longitude, or, more exactly in the case of the occultations, to a change in t, are substantially included. The earlier justification for this plan arose from the discussion of the principal elliptic term in the Greenwich observations for 1750–1910, where it appeared that its oscillations corresponded with those of the mean longitude (*M.N.*, **74**, 405). The later justification arises from the evidence that the chief differences between the gravitational orbit and the observed positions are due to changes in our measure of time.

6. Table I contains a summary of the results for the 111 lunations included in the nine years, lunation 1 beginning with the new Moon of 1923 January 16.

The equation of condition satisfied by each occultation is

$$ax + by = d.$$

Here x is the residual in longitude after 7″·00 for the years 1923–28 and 6″·00 for 1929–31 have been added to the tabular mean longitudes; $-y$ is the residual in latitude above the tabular instantaneous plane of the orbit. The cosine and sine of $\chi - \rho$, the position angle with respect to the orbit, of the radius from the Moon's centre to the occulted star, are denoted by a, b respectively, so that $a^2 + b^2 = 1$. Finally, d, usually denoted by $\sigma' - \sigma$, is the difference between the observed and tabular values of this radius.

In Table I:

Col. 1 gives the number of the lunation; the years 1925, 1928, 1931 each contain 13 lunations, and the remaining years 12 lunations.

Cols. 2–6 give the coefficients $[aa]$, $[ab]$, $[bb]$, $[ad]$, $[bd]$ of the normal equations for each lunation and the next two columns the values, $x = \delta\lambda_l$, $y = -\delta\beta_l$ deduced from them.

In Table I_q are given $x = \delta\lambda_q$, $y = -\delta\beta_q$ obtained by adding the normal equations for each three consecutive lunations, solving and placing the result opposite the middle one. These are thus overlapping quarterly solutions.

The residuals R_λ given in the last column of Table I_q show the extent to which the quarterly values are satisfied when certain observed systematic

changes have been subtracted from $\delta\lambda_q$. They are obtained as follows: The mean change per lunation, namely, 0″·028, over the whole nine years is subtracted from $\delta\lambda_q$; the result is shown in the upper curve of fig. 1. These values for each year are then analysed for a constant and an annual harmonic term, and the amounts so obtained are subtracted from $\delta\lambda_q + 0''{\cdot}028l$ to give R_λ.

TABLE I

Normal Equations

Lun.	[*aa*]	[*ab*]	[*bb*]	[*ad*]	[*bd*]	$\delta\lambda_l - 7''$	$-\delta\beta_l$
[1923]				″	″	″	″
1	15·04	+ ·59	9·96	+ 10·9	− ·5	+ ·73	− ·09
2	2·72	+ 1·63	1·28	+ 5·4	+ 3·1	+2·25	− ·45
3	12·53	− 6·69	12·47	+ 19·6	−12·0	+1·47	− ·17
4	16·25	+ 3·94	4·75	+ 16·3	+ 3·0	+1·06	− ·25
5	15·68	− 2·64	6·32	+ 13·0	− 3·5	+ ·79	− ·22
6	4·17	+ 1·01	1·83	+ 8·6	+ 3·6	+1·83	+ ·96
7	8·60	+ 2·61	2·40	+ 11·0	+ 4·5	+1·06	+ ·72
8	14·50	+ 3·32	10·50	+ 16·4	+10·2	+ ·98	+ ·66
9	18·30	− 2·46	11·70	+ 6·3	+11·3	+ ·49	+1·08
10	30·87	− 8·57	19·13	+ 21·1	+20·9	+1·13	+1·61
11	4·65	− 1·50	1·35	+ 1·3	− ·3	+ ·32	+ ·14
12	7·38	+ ·34	1·62	− 1·2	+ 1·7	− ·21	+1·09
[1924]							
13	15·79	− 8·34	11·21	+ 14·3	− 6·5	+ ·98	+ ·16
14	15·10	− 3·05	6·90	+ 16·7	− 3·7	+1·10	− ·06
15	1·54	+ ·67	2·46	+ 2·6	+ 1·0	+1·66	− ·04
16	20·37	− 1·32	7·63	+ 23·4	− 1·8	+1·15	− ·03
17	10·51	+ 2·95	3·49	+ 14·5	+ 4·6	+1·33	+ ·21
18	8·65	+ ·59	3·35	+ 7·4	+ ·7	+ ·86	+ ·05
19	5·54	+ 1·39	1·46	+ 4·7	+ 1·7	+ ·73	+ ·47
20	13·67	− 2·09	9·33	− ·4	+ 6·7	+ ·08	+ ·90
21	3·16	+ 1·43	·84	+ ·3	+ ·6	− ·96	+2·4
22	9·22	− 4·42	2·78	+ 2·1	− ·1	+ ·85	+1·29
23	12·00	− 2·70	4·00	+ 4·8	+ 1·2	+ ·55	+ ·68
24	2·71	+ ·19	2·29	+ 3·8	+ ·4	+1·41	+ ·04
[1925]							
25	11·05	+ ·44	2·95	+ 1·3	− 1·2	+ ·13	− ·43
26	34·0	+ 4·53	14·96	+ 19·8	+15·3	+ ·46	+ ·88
27	6·53	+ 1·07	1·47	+ 4·1	+ 1·0	+ ·59	+ ·25
28	6·41	+ ·60	2·59	− 4·2	− 5·3	− ·47	−1·94
29	5·18	+ ·80	1·82	+ 5·9	+ ·2	+1·20	− ·34
30	3·41	− ·94	1·59	+ 5·3	− 1·7	+1·50	− ·18
31	5·18	− ·90	1·82	+ 6·0	+ 1·1	+1·38	+1·29
32	7·82	− ·74	1·18	+ 1·3	− 2·1	·00	−1·77
33	9·47	+ 1·99	5·53	− 1·8	− 1·1	− ·16	− ·14
34	8·66	− ·18	3·34	+ 1·2	− 2·6	+ ·12	− ·77
35	9·87	− ·85	3·13	− ·2	− ·2	− ·03	− ·10
36	4·84	− ·62	1·16	+ 2·9	+ 1·0	+ ·76	+1·27
37	4·97	− 1·52	4·03	+ 1·1	+ 2·9	+ ·50	+ ·88

Lun.	[*aa*]	[*ab*]	[*bb*]	[*ad*]	[*bd*]	$\delta\lambda_1 - 7''$	$-\delta\beta_1$
[1926]				″	″	″	″
38	6·69	− ·08	·36	+ 3·9	·0	+ ·60	+ ·15
39	9·16	+ 1·58	2·82	+ 2·9	+ 2·8	+ ·16	+ ·89
40	10·37	− ·43	2·62	+ 2·8	+ ·7	+ ·28	+ ·31
41	12·93	− ·32	5·13	+ 13·0	+ ·5	+ 1·01	+ ·16
42	4·77	+ ·56	2·20	+ 3·3	+ 1·0	+ ·67	+ ·27
43	6·40	+ ·11	1·58	− ·7	+ ·1	− ·11	+ ·06
44	10·11	− 1·63	3·88	− 6·4	+ ·8	− ·65	+ ·08
45	6·22	+ 1·95	1·75	+ 3·1	+ ·1	+ ·74	− ·74
46	11·51	+ 2·18	4·48	·0	+ 4·6	− ·22	+ ·91
47	9·22	− 1·51	3·71	+ ·1	− ·3	·00	− ·08
48	11·80	− ·49	5·17	− ·1	+ 3·2	+ ·02	+ ·61
49	5·35	− ·93	1·67	+ ·2	− 1·1	− ·08	− ·72
[1927]							
50	14·28	+ ·37	2·76	− ·3	+ 4·2	− ·06	+ 1·53
51	30·67	+ ·01	13·18	+ 6·5	+ 17·5	+ ·21	+ 1·32
52	16·49	+ ·99	8·49	+ 7·3	− 4·9	+ ·48	− ·60
53	25·54	+ ·67	7·49	+ 11·9	+ 9·8	+ ·43	+ 1·27
54	13·92	+ 3·24	4·07	+ 2·9	+ ·8	+ ·20	+ ·04
55	16·89	+ ·95	6·07	− 8·3	·0	− ·50	+ ·08
56	21·58	− 2·67	8·36	− 17·5	+ 5·8	− ·76	+ ·46
57	14·71	+ 1·81	4·34	− 4·8	+ ·1	− ·35	+ ·17
58	20·96	− ·94	4·05	− 4·7	+ 3·4	− ·19	+ ·79
59	22·94	− 5·07	13·06	− 16·6	+ 15·5	− ·50	+ ·99
60	31·98	− 7·90	7·99	− 18·7	+ 4·1	− ·61	− ·08
61	40·89	− 4·00	14·19	− 12·5	+ 12·5	− ·23	+ ·82
[1928]							
62	24·24	− 6·34	14·62	− 3·0	+ 4·8	− ·04	+ ·31
63	30·51	− 3·45	8·52	− 18·4	+ 8·5	− ·51	+ ·79
64	65·58	− 4·92	19·40	− 19·6	+ 11·9	− ·26	+ ·55
65	38·03	+ 3·46	13·98	− 9·2	+ 14·8	− ·35	+ 1·14
66	67·16	− 3·00	37·73	− 22·4	+ 50·9	− ·27	+ 1·33
67	26·30	− ·27	10·70	− 15·3	+ 18·6	− ·56	+ 1·72
68	25·71	− ·76	7·25	− 24·3	+ 3·8	− ·93	+ ·43
69	25·56	+ ·58	9·36	− 24·4	+ 8·3	− ·98	+ ·95
70	35·80	+ ·13	8·16	− 47·3	− 2·3	− 1·32	− ·26
71	23·88	− 2·62	8·04	− 40·8	+ 3·1	− 1·73	− ·17
72	32·45	− 4·01	12·45	− 29·5	+ 8·5	− ·86	+ ·40
73	39·92	− 2·89	17·01	− 56·9	+ 13·1	− 1·39	+ ·53
74	43·70	− 1·27	13·23	− 29·0	+ 6·8	− ·65	+ ·45
[1929]						$\delta\lambda_1 - 6''$	
75	39·74	− ·26	7·34	− 3·9	− 1·5	− ·10	− ·21
76	58·37	− 6·42	18·62	+ 17·9	+ 12·1	+ ·39	+ ·78
77	82·32	+ 7·37	29·63	+ 41·7	+ 11·9	+ ·48	+ ·28
78	73·34	− 9·28	35·54	− ·8	+ 43·3	+ ·15	+ 1·25
79	39·51	+ 3·44	12·52	+ 6·8	+ 24·7	·00	+ 1·97
80	21·23	− ·38	9·78	·0	+ 8·8	+ ·02	+ ·90
81	23·59	− 4·63	13·34	− 12·6	+ 23·0	− ·21	+ 1·66
82	27·74	− 5·49	14·31	− 16·0	+ 18·0	− ·35	+ 1·12
83	30·58	− 6·03	12·40	− 17·1	+ 11·4	− ·42	+ ·72
84	20·57	− 5·07	9·40	− 2·9	− 1·0	− ·19	− ·21
85	17·95	− 4·03	7·04	− 16·1	+ 12·2	− ·58	+ 1·40
86	11·50	+ ·21	8·44	− 7·0	+ 1·3	− ·61	+ ·17

Lun.	[*aa*]	[*ab*]	[*bb*]	[*ad*]	[*bd*]	$\delta\lambda_l - 6''$	$-\delta\beta_l$
[1930]				″	″	″	″
87	27·91	− ·52	13·03	− 17·0	− 5·5	− ·62	− ·45
88	22·38	− 1·79	12·61	− 6·1	+11·8	− ·20	+ ·91
89	47·06	+ 2·98	12·90	− 8·4	+ 4·7	− ·20	+ ·41
90	30·28	+ 1·79	11·70	− 9·5	+ 1·1	− ·32	+ ·14
91	28·65	+ 9·21	14·34	+ 17·3	+16·6	+ ·29	+ ·97
92	45·24	+ 6·86	17·77	+ 31·7	+21·8	+ ·55	+1·02
93	24·30	+ 3·73	12·74	+ 16·3	+18·7	+ ·47	+1·33
94	24·91	− 1·10	8·04	− ·4	+11·4	+ ·05	+1·42
95	17·29	+ ·01	5·71	− 5·4	+ 6·2	− ·31	+1·09
96	55·81	+ 6·66	25·17	− 36·0	+25·9	− ·79	+1·24
97	44·38	+ 2·12	14·61	− 49·2	+ ·3	−1·12	+ ·28
98	54·02	+11·61	25·98	− 57·1	+10·6	−1·27	+ ·97
[1931]							
99	16·14	− 3·41	9·86	− 14·1	+ 8·7	− ·74	+ ·63
100	22·18	− ·68	18·82	− 17·2	+16·4	− ·75	+ ·85
1	42·50	+ 2·08	18·50	− 29·9	+ 9·2	− ·73	+ ·56
2	56·70	+ 4·88	24·30	− 9·8	+13·5	− ·22	+ ·60
3	48·15	+ 3·02	19·85	− 25·3	+ 5·0	− ·55	+ ·34
4	29·91	+ 4·64	11·09	− 16·6	− 6·0	− ·50	− ·33
5	26·88	+ 5·21	16·12	+ ·1	+ ·1	·00	+ ·01
6	40·72	− 6·44	18·28	− 38·2	+25·0	− ·76	+1·09
7	26·58	+ 3·57	11·42	− 14·1	− 1·0	− ·54	+ ·08
8	41·60	− 5·52	13·40	− 67·6	+18·2	−1·53	+ ·73
9	79·73	− 6·43	28·27	−103·4	+41·0	−1·20	+1·18
110	44·67	− 5·48	18·33	− 61·2	+15·1	−1·32	+ ·43
1	82·90	−18·10	38·10	−140·3	+54·6	−1·54	+ ·70

7. If we could properly judge of the accuracy of a quarterly mean from the residuals R_λ, its average probable error would be considerably less than 0″·1. The actual relative weights of the determination can be directly obtained from Table I; since in most cases the cross term is small, the approximate weights of $\delta\lambda_l$, $\delta\beta_l$ are shown by the coefficients [*aa*], [*bb*] respectively, and those of $\delta\lambda_q$, $\delta\beta_q$ by the corresponding sums in threes of these coefficients. On the basis of the assignment of a probable error of 1″ to each value of *d*, they may be converted into probable errors by means of the formula

$$\epsilon = \pm\, 1'' \,.\, w^{-\frac{1}{2}}.$$

8. In order properly to form the annual means, it is necessary to consider the known errors of the observations and reductions.

The observer records only the time of the occultation and the name of the star occulted. An error of 1^s in the time corresponds to an average error of 0″·5 in $\delta\lambda$. This error has been treated as accidental. Although it may result in an average error of one or two tenths of a second of arc in $\delta\lambda$, this average will differ little from year to year and thus merely give a constant addition to $\delta\lambda$ which will be included in the comparison with the meridian observations.

9. With large numbers of observations the errors due to irregularities of the Moon's limb give an average error in a year which should be constant and therefore similar to the time error. The limb, however, exhibits

TABLE I_a

Lun.	$\delta\lambda_a - 7''$	$-\delta\beta_a$	R_λ	Lun.	$\delta\lambda_a - 7''$	$-\delta\beta_a$	R_λ	Lun.	$\delta\lambda_a - 6''$	$-\delta\beta_a$	R_λ
	″	″	″		″	″	″		″	″	″
1				38	+ ·38	+ ·79	−·12	75	+ ·23	+ ·45	+·11
2	+1·16	− ·18	+·03	39	+ ·35	+ ·54	−·20	76	+ ·31	+ ·40	+·11
3	+1·30	− ·24	+·14	40	+ ·58	+ ·33	+·04	77	+ ·30	+ ·83	+·05
4	+1·06	− ·29	−·11	41	+ ·68	+ ·23	+·22	78	+ ·24	+1·02	−·01
5	+1·04	+ ·05	−·10	42	+ ·64	+ ·16	+·32	79	+ ·11	+1·34	−·05
6	+1·13	+ ·33	+·05	43	− ·14	+ ·24	−·32	80	− ·04	+1·58	−·05
7	+1·14	+ ·71	+·14	44	− ·18	+ ·15	−·21	81	− ·21	+1·27	−·04
8	+ ·73	+ ·96	−·18	45	− ·17	+ ·58	−·09	82	− ·33	+1·18	·00
9	+ ·83	+1·18	·00	46	+ ·08	+ ·42	+·21	83	− ·32	+ ·64	+·11
10	+ ·84	+1·32	+·08	47	·00	+ ·56	+·11	84	− ·40	+ ·57	+·08
11	+ ·81	+1·37	+·08	48	+ ·03	+ ·18	+·05	85	− ·46	+ ·34	−·01
12	+ ·51	·00	−·22	49	+ ·02	+ ·66	−·05	86	− ·69	+ ·18	−·33
13	+ ·78	·00	−·26	50	+ ·14	+1·18	−·11	87	− ·48	+ ·20	+·20
14	+1·07	+ ·11	−·11	51	+ ·21	+ ·65	−·09	88	− ·33	+ ·29	+·14
15	+1·15	− ·01	−·07	52	+ ·34	+ ·75	+·07	89	− ·26	+ ·50	−·08
16	+1·25	+ ·07	+·09	53	+ ·38	+ ·19	+·22	90	− ·09	+ ·61	−·19
17	+1·14	+ ·07	+·14	54	+ ·07	+ ·59	+·09	91	+ ·24	+ ·80	−·06
18	+1·03	+ ·23	+·24	55	− ·45	+ ·40	−·22	92	+ ·45	+1·08	+·10
19	+ ·42	+ ·65	−·13	56	− ·58	+ ·32	−·18	93	+ ·38	+1·25	+·14
20	+ ·18	+ ·77	−·17	57	− ·46	+ ·51	+·05	94	+ ·10	+1·36	+·12
21	+ ·20	+ ·63	−·05	58	− ·39	+ ·81	+·14	95	− ·49	+1·29	−·12
22	+ ·42	+ ·54	+·16	59	− ·40	+ ·69	+·07	96	− ·84	+ ·87	−·03
23	+ ·64	+ ·65	+·28	60	− ·37	+ ·73	−·02	97	−1·04	+ ·88	−·07
24	+ ·40	+ ·13	−·12	61	− ·27	+ ·45	−·07	98	−1·11	+ ·42	−·04
25	+ ·46	+ ·60	·00	62	− ·27	+ ·59	+·22	99	−1·02	+ ·79	−·10
26	+ ·41	+ ·65	−·11	63	− ·28	+ ·49	+·02	100	− ·74	+ ·70	+·03
27	+ ·36	+ ·46	−·24	64	− ·32	+ ·80	−·11	101	− ·50	+ ·69	+·09
28	+ ·44	− ·89	−·23	65	− ·27	+1·07	−·05	102	− ·48	+ ·52	−·04
29	+ ·50	−1·17	−·19	66	− ·36	+1·35	−·01	103	− ·41	+ ·32	−·05
30	+1·26	+ ·17	+·01	67	− ·48	+1·28	+·12	104	− ·41	+ ·09	−·01
31	+ ·74	− ·17	+·16	68	− ·82	+1·11	+·03	105	− ·59	+ ·46	−·03
32	+ ·25	− ·26	−·18	69	−1·12	+ ·39	−·03	106	− ·57	+ ·55	+·17
33	+ ·05	− ·58	−·25	70	−1·31	+ ·26	−·07	107	−1·04	+ ·78	−·06
34	− ·02	− ·32	−·20	71	−1·28	+ ·03	+·01	108	−1·21	+ ·91	−·03
35	+ ·15	− ·21	+·05	72	−1·29	+ ·33	−·07	109	−1·31	+ ·86	·00
36	+ ·27	+ ·54	+·18	73	− ·96	+ ·48	+·06	110	−1·28	+ ·85	+·05
37	+ ·61	+ ·96	+·50	74	−1·00	+ ·42	−·18	111			

sensible deviations from a circle over considerable arcs as well as local irregularities, and these alter with the libration, which depends mainly on the difference between the true and mean longitudes of the Moon. There is thus a possibility of a systematic deviation with a principal period of an anomalistic month: in the means by lunations this would appear as a fourteen-month term in the longitude, since the occultations are usually confined to less than one-third of the lunation. The fact that there is little evidence of any such term in $\delta\lambda$ appears to show that the systematic effect must be small. (See E. W. Brown, *A.J.*, 901; D. Brouwer, *A.J.*, 955, and previous work to which reference is there made.) A rough estimate of the inequalities of the distribution of the occulted stars over the limb can be gathered from the sign and magnitude of the ratio of $[ab]$ to $[aa]$ in Tables I, II, but it gives no information as to the effect on the concluded values of x, y. There is probably enough material in the published results for the

occultations to obtain these systematic errors with sufficient accuracy if it were classified according to values of $\chi - \rho$ and $\lambda - M$, and according to values of $\chi - \rho$ and β, where λ and M are the true and mean longitudes of the Moon and β is its latitude. But the whole of the material must be so corrected in order to take care of the average change of radius which any such system will certainly produce.

10. The most serious systematic effect is probably that which arises from the adopted positions of the occulted stars. For the bright stars, the positions given in the almanacs are obtained from the Washington Zodiacal Catalogue. For the fainter stars, the positions have in general been obtained by the observer or computer from various sources and no attempt has been made to reduce them to a uniform system. For the faint stars reduced by the members of the B.A.A. and A.A.V.S.O., most of the positions have lately been obtained from the Dudley Observatory. As is well known, the catalogues contain errors varying with α, δ. The effect on the occultations is discussed below (§ 21) in connection with the annual term in $\delta\lambda$.

11. Errors of computation in the reductions appear to be negligible as far as any systematic effect is concerned. The error in the formula used in the earlier years—a wrong sign for the coefficient of $\delta\beta$—has been corrected ; but in any case it had no effect on the concluded value of $\delta\lambda$. The great majority of the computations have been done in duplicate, with the use of the formulæ of Innes given in *A.J.*, 855. The summaries of the present paper have been tested by recomputation and by applying obvious checks.

12. The method used here for finding the observed annual values of $\delta\lambda$, $\delta\beta$ has been adopted in consequence of the accidental and systematic errors noted above and to bring out clearly the real deviations shown by the observations ; it differs somewhat from that used by Dr. Brouwer and myself in our annual summaries. In the earlier years there are a number of cases of lunations in which the number of occultations is so small that the accidental errors give abnormal values to $\delta\lambda$, $\delta\beta$, especially to the latter. The formation of quarterly normal equations in great part avoids this difficulty, and the theoretical effect on the annual and oscillating changes is practically insensible. In the later years, where the occultations are numerous, this procedure, as against the use of the results from the separate lunations, is unnecessary, but for the sake of uniformity the whole series has been treated in this manner. In the use of the quarterly means to obtain results for any given year, the last lunation of the previous year and the first of the following year enter with weight 1/3, the first and last lunations of the year with weight 2/3, as compared with unit weight for each of the remaining lunations. Results obtained by treating each quarterly mean, each lunation mean, and each observation, as of unit weight, are denoted by the suffixes q, l, o respectively.

The Annual Means.

13. The annual means formed in three different ways are shown in Tables II, III.

Table II gives the annual normal equations, treating each occultation as

of unit weight, with their solutions $\delta\lambda_o$, $\delta\beta_o$. Since the cross terms are small, the weights of $\delta\lambda_o$, $\delta\beta_o$ are respectively shown by the coefficients [*aa*], [*bb*], the conversion into probable errors being made as before by the formula given in No. 7.

In Table III are shown the annual means derived from the results of Tables I, I_q, II. The averages by lunations, $\delta\lambda_l$, $\delta\beta_l$, are those given by Dr. Brouwer and myself in our annual summaries for the years 1927–31, but the 0″·08 there added to conform with the results for 1927 has not been applied here.

TABLE II

Annual Normal Equations

Lun.	[*aa*]	[*ab*]	[*bb*]	[*ad*]	[*bd*]	$\delta\lambda_o$		$-\delta\beta_o$	
				″	″	″		″	
1– 12	150·7	− 8·4	83·3	+128·7	+ 42·0	+7·89	±·08	+·59	±·11
13– 24	118·3	−14·7	55·7	+ 94·2	+ 4·8	+7·84	9	+·31	14
25– 37	117·4	+ 3·7	45·6	+ 42·7	+ 7·3	+7·36	9	+·13	15
38– 49	104·6	+ 1·0	35·4	+ 22·1	+ 12·4	+7·21	10	+·34	17
50– 61	270·9	−12·5	94·1	− 54·8	+ 68·8	+6·83	6	+·71	10
62– 74	479·1	−25·4	180·9	−340·1	+150·8	+6·33	5	+·74	8
75– 86	446·5	−30·6	178·5	− 10·0	+164·2	+6·03	5	+·93	8
87– 98	422·3	+41·6	174·7	−123·8	+123·6	+5·63	5	+·80	8
99–111	558·7	−22·7	246·3	−537·6	+199·8	+5·07	4	+·73	6

TABLE III

Results from Occultations

	$\delta\lambda_q$		$\delta\lambda_l - \delta\lambda_q$	$\delta\lambda_o - \delta\lambda_q$	$-\delta\beta_q$	$-\delta\beta_l + \delta\beta_q$	$-\delta\beta_o + \delta\beta_q$
	″		″	″	″	″	″
1923	+7·96		+0·03	−0·07	+0·47	−0·05	+·012
		−24					
4	7·72		+ 9	+ 12	+ ·32	+ 19	− 1
		30					
5	7·42		+ 4	− 6	− ·02	− 6	+ 15
		23					
6	7·19		+ 1	+ 2	+ ·40	− 24	− 6
		34					
7	6·85		− 1	− 2	+ ·61	− 4	+ 10
		60					
8	6·25		− 1	+ 8	+ ·66	− 3	+ 8
		35					
9	5·90		− 2	+ 13	+ ·82	0	+ 11
		19					
30	5·71		0	− 8	+ ·79	− 2	+ 1
		51					
1	5·20		0	− 13	+ ·63	− 10	+ 10

The differences between $\delta\lambda_q$, $\delta\lambda_l$ are rather smaller than might be expected, only one being as large as 0″·09, the remainder being 0″·04 or less. Those between $\delta\lambda_o$ and $\delta\lambda_q$ or $\delta\lambda_l$ are larger and are evidently due to unequal distribution, combined with the annual term; they are most marked in the later years where the annual term is most prominent, as might be expected from the distribution.

The chief value of the quarterly means is in the determination of $\delta\beta$ in the earlier years, the differences between $\delta\beta_q$, $\delta\beta_l$, $\delta\beta_o$ in the later years are satisfactory. The evidence given below of a fourteen-month oscillation which differs considerably from a single harmonic term makes the determination of an annual value for $\delta\beta$ more doubtful than would be the case if such a term did not exist. It will be seen below that the apparent oscillation in the annual means of $\delta\beta$ is partly, if not wholly, due to a distribution effect of the fourteen-month oscillation.

14. Table IV gives a comparison of the annual values of $\delta\lambda_q$ with those obtained from the Greenwich and Washington meridian observations, as given in the publications referred to in No. 4. The Greenwich limb observations (*GL*) are kept separate from those of Mösting A (*GA*). In forming these means, I assume that each observation has been given unit weight. As there appears to be little evidence of an annual term (No. 19), it is unnecessary to form means by lunations.

15. A final mean (*M*) is derived from the four sets by giving equal weights to the results from the occultations, to those from the Greenwich limb, to those from Washington, and half weight to those of Mösting A, since the number of observations of the crater is, in general, not much greater than half that of the limb. Owing to the numbers of observations, the values of $\delta\lambda_q$ for the later years should perhaps receive higher weights, but until the systematic errors of the latter are cleared up, I prefer to retain the simple weights assigned above.

The means for the nine years are shown in the last line : these are to be regarded as constant differences to be applied to the respective sets for mutual

Table IV

Annual Means of $\delta\lambda$

	$\delta\lambda_q$	*GL*	*W*	*GA*	*M*	
	″	″	″	″	″	
1923	+7·96	+6·99	+6·91	+7·47	+7·18	
						− 4
4	7·72	7·21	6·94	7·14	7·14	
						48
5	7·42	6·61	6·39	6·71	6·66	
						22
6	7·19	6·40	6·11	6·55	6·44	
						37
7	6·85	6·20	5·52	6·24	6·07	
						53
8	6·25	5·51	5·13	5·88	5·54	
						26
9	5·90	5·20	4·90	5·89	5·28	
						27
30	5·71	5·08	4·48	5·40	5·01	
						36
1	5·20	4·76	4·34	4·87	4·65	
Means	6·69	6·00	5·64	6·24	6·00	

	$\delta\lambda_q - 0''\cdot69 - M$	$GL - M$	$W + 0''\cdot36 - M$	$GA - 0''\cdot24 - M$
	″	″	″	″
1923	+0·09	−0·19	+0·09	+0·05
4	− ·11	+ ·07	+ ·16	− ·24
5	+ ·07	− ·05	+ ·09	− ·19
6	+ ·06	− ·04	+ ·03	− ·13
7	+ ·09	+ ·13	− ·19	− ·07
8	+ ·02	− ·03	− ·05	+ ·10
9	− ·07	− ·08	− ·02	+ ·37
30	+ ·01	+ ·07	− ·17	+ ·15
1	− ·14	+ ·11	+ ·05	− ·02

comparisons. The set M is formed with the constant of the Greenwich limb observations as the basis, since the tabular constants have been derived mainly from the Greenwich meridian observations of the limb.

The last four columns of Table IV show the differences from the mean of the four sets, after the application of the constants. On the whole they are satisfactory and indicate that, as far as the observations are concerned, we do, in fact, know the mean annual apparent motion of the Moon with a probable error of less than $0''\cdot1$.

The deviations from uniform motion are shown by the column of differences of M, the mean difference being $0''\cdot32$. The changes at 1923–24, and at 1927–28 appear to be real; the remaining deviations from uniform motion being more doubtful.

16. Table V gives the annual differences from pure gravitation theory, now ascribed to changes in our measure of time.

The column $-\delta T$ is the reduction (tabulated in *A.J.*, 799) of the tabular value of the mean longitude to that deduced by Fotheringham, including a change in the great empirical term; G.E.T. is the tabulation of the new empirical term, to be subtracted when the non-gravitational value is needed; the column M is the mean value of the observed minus tabular deviation deduced in Table IV; $O - Th$ gives the required non-gravitational deviation.

The last columns of the table denoted by Δ and $+0''\cdot42y$ show the deviations of the annual differences from a uniform rate during the nine years. Those less than $0''\cdot10$ may properly be ascribed to accidental errors; it is not possible to make a statement more definite than that in the last paragraph but one concerning the others. The present indications for the year 1932 point to a change from 1931 about the same as that between 1930–31.

On the basis of this paper—the same as that adopted earlier *—the greatest negative deviation of the Moon from its gravitational orbit, so far as it is known, occurred about 1897 with a value $-15''\cdot7$. If the present annual rate of $-0''\cdot4$ is maintained, a similar value will be reached in 1935. As, however, I have pointed out on several previous occasions, the rate is apt to change suddenly after being nearly uniform for a number of years, so that little stress can be laid on any prediction. It is, however, the best that can be done for eclipse predictions until observational evidence of a change is available. To such predictions the differences investigated by Brouwer (*A.J.*, 934) between the occultation and eclipse results should be applied.

* *Trans. Yale Obs.*, 3, 218.

Table V

Deviations from Gravitational Theory

	M	$-\delta T$	$-$G.E.T.	$O-Th$	Δ	$+0''\cdot42y$
	″	″	″	″		
1923	+7·18	−4·42	−13·53	−10·77		
					−15	+27
4	7·14	4·57	13·49	10·92		
					60	−18
5	6·66	4·73	13·45	11·52		
					32	+10
6	6·44	4·89	13·39	11·84		
					48	− 6
7	6·07	5·06	13·33	12·32		
					62	−20
8	5·54	5·22	13·26	12·94		
					35	+ 7
9	5·28	5·39	13·18	13·29		
					36	+ 6
30	5·01	5·56	13·10	13·65		
					45	− 3
1	4·65	5·74	13·01	14·10		
2		5·92	12·91			
3		6·10	12·80			
4		6·28	12·69			
5		6·47	12·57			

17. Table VI shows the annual means of $-\delta\beta$ derived from the four sets of observations. As might be expected from the known systematic irregularities of the Moon's limbs, the means of the meridian observations differ from that of the occultations. If, however, we use the last five years only when the occultations are very numerous, the difference is much smaller. At present the evidence points to a correction to the tabular latitude of $-0''\cdot70$ rather than the $-0''\cdot50$ which was deduced from the material before 1923,* and this new value appears to be the best for eclipse predictions for the present.

Table VI

Annual Values of $-\delta\beta$

	$-\delta\beta_q$	GL	W	GA
	″	″	″	″
1923	0·47	0·56	0·59	0·63
4	·32	·82	·63	·59
5	− ·02	·87	·73	·75
6	·40	·68	·76	·92
7	·61	·70	·73	·91
8	·66	·93	·87	·66
9	·82	·78	·75	·80
30	·79	·93	·72	·83
1	·63	·82	·86	·71
Means	·52	·79	·74	·76

* See *Publ. of Naval Obs.*, **13**, 1932, App. iv, p. 214.

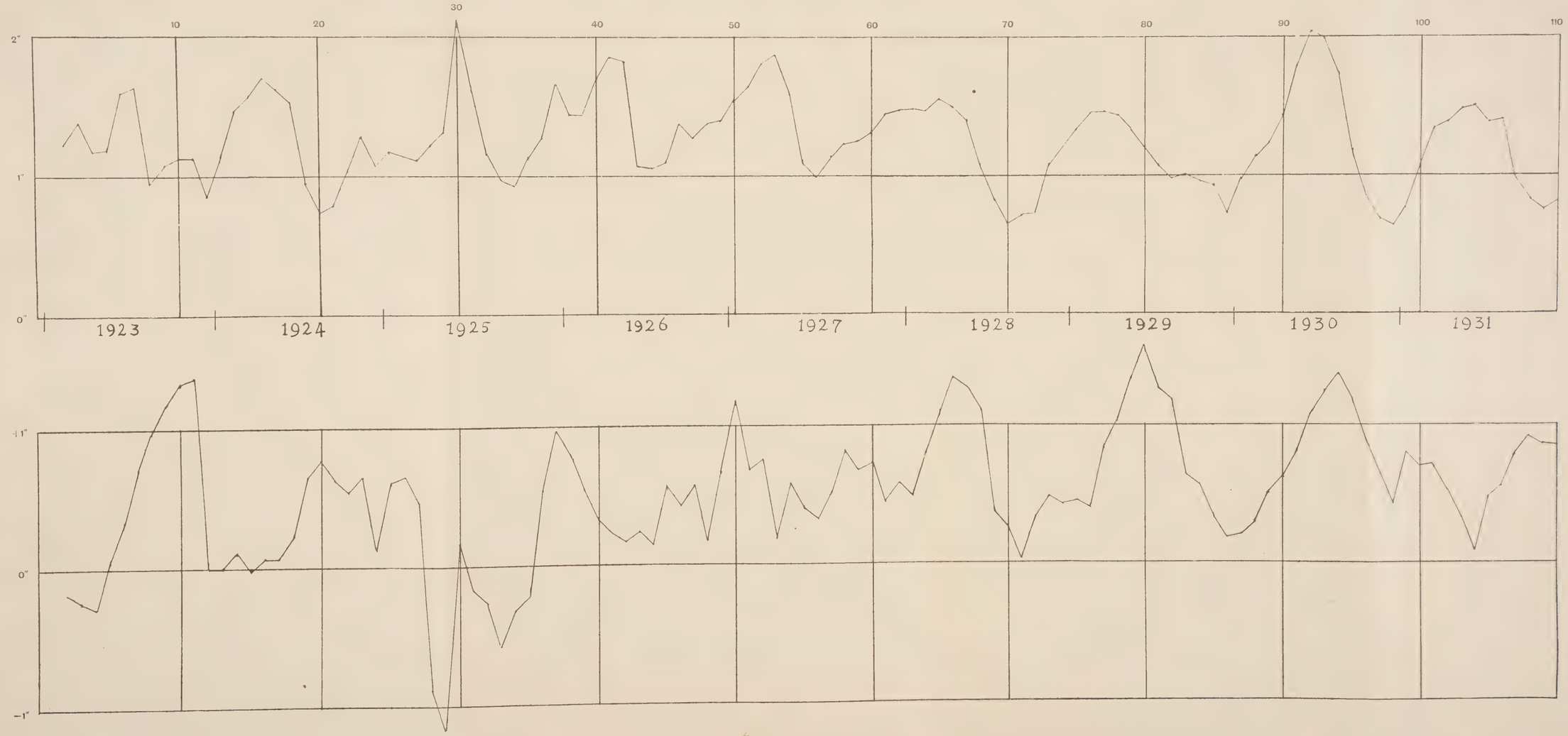

Fig. 1.—Upper curve, $\delta\lambda_0 - 7''{\cdot}00 + 0''{\cdot}028l$; lower curve, $-\delta\beta_0$. Abscissæ show number of lunation and year.

E. W. Brown, The Motion of the Moon, 1923–1931.

It is doubtful whether any of the differences between the annual means are real. As pointed out above, the oscillation in the occultation results is, at least in part, a distribution effect of the fourteen-month term; there is little evidence of any such change in the meridian observations. The question must await future observations for settlement.

The Annual Term in Longitude

18. The upper curve of the diagram shows how this term has persisted throughout nearly the whole extent of the interval. Curiously enough there is little or no evidence of it in 1923, the year in which Wood pointed out its existence, but it should be remembered that he plotted the observations according to the right ascension of the Moon, and in the diagram they are plotted according to lunation, and it may happen that the distribution in that year was such as to blot out the term in the latter method of discussion. The question can, of course, be settled from an examination of the material; in this paper, however, I limit myself to the results which can be deduced from the means by lunations or by years.

The diagram shows that the period is so close to a year that analysis of each year for an annual period will give a coefficient and phase not sensibly affected by an error in the period. Table VII gives the results of such an analysis. The columns s, c are the calculated coefficients of the sine and cosine, reckoning the angle from the first lunation of the year in each case; C is the concluded coefficient of the harmonic term and A the angle corrected to the beginning of the year on the assumption that the average time of the occultations utilised was five days before full Moon. The probable errors are estimated as in the earlier part of the discussion. Before analysis the mean change, $0''.028l$, has been added, and the mean value of the resulting numbers subtracted. This procedure, admittedly somewhat rough, should be sufficiently accurate to indicate the lines on which a more exact analysis could be carried out.

Table VII

Annual Term in $\delta\lambda_a$

	s	c	C	A	ϵ_z, ϵ_s
	$''$	$''$	$''$	$^\circ$	
1923	+0·12	−0·04	0·13	135	± 0·12
4	+ ·37	+ ·16	·40	83	·14
5	+ ·16	− ·14	·21	144	·14
6	+ ·20	+ ·15	·25	76	·14
7	+ ·23	+ ·24	·33	56	·09
8	+ ·45	+ ·10	·46	81	·08
9	+ ·28	+ ·05	·28	100	·07
30	+ ·32	− ·54	·63	158	·07
1	+ ·29	− ·28	·40	133	·06
1923–6	+ ·21	+ ·03	·21	82	
7–1	+ ·31	− ·09	·32	106	

The probable errors indicate that the phases for the years 1923–26 have little significance, yet they vary less than might have been expected. The last two lines in which the results are combined into two groups of four and five years respectively show that there is no significant change in the period during the interval. The column R_λ in Table I_a shows the residuals after this harmonic term is subtracted. There is evidence of a harmonic oscillation with a period of half a year. As they stand, however, they are smaller on the whole than would be expected from the assigned probable errors.

19. Table VIII gives the results of a corresponding analysis for the two groups, A 1923–26 and B 1927–31, for the occultations and Washington meridian observations and 1927–30 for the Greenwich meridian observations; in all cases, those for each lunation are treated as having the same weight.

TABLE VIII

Mean Monthly Changes of $\delta\lambda$. *Units of* 0″·01

	1923–26				1927–30, 31			
	$\delta\lambda_a$	*GL*	*W*	*GA*	$\delta\lambda_a$	*GL*	*W*	*GA*
1	+ 6	−16	−11	+16	− 2	+ 2	+26	+ 5
2	+ 5	+11	−19	+19	+12	+45	+25	+68
3	+18	+45	+ 4	+39	+23	+75	+30	+25
4	+21	+24	+ 8	−39	+30	− 8	+13	+ 2
5	+22	− 8	+ 7	+34	+30	+ 5	−31	−58
6	+23	−38	− 8	−17	+22	−30	−61	−10
7	− 3	+25	−13	−28	+ 7	+ 3	−17	+28
8	−28	+31	−14	− 7	− 2	−40	−56	− 5
9	−21	−14	+14	−37	−23	−30	0	+ 2
10	−17	− 9	+33	+ 4	−29	−42	−13	+25
11	− 4	−11	−22	+ 2	−35	−10	+45	− 7
12	−12	−23	+10	+17	−29	+20	+36	−20

The similarity of the two groups for the occultations is again evident, and there is practically no resemblance between them and the meridian observations. There is a slight effect in the Washington groups (also pointed out by Morgan, *loc. cit.*, No. 17), but it is not the same in the two groups and its phase differs considerably from that of the occultations. Similar remarks may be made concerning the Greenwich observations. It is concluded that the marked annual term present in the occultations is peculiar to that class of observations.

The Fourteen-month Oscillation in the Latitude

20. The lower curve in the diagram shows a marked oscillation of the latitude during the years 1927–31, with a period of some fourteen months. The evidence for its existence in the earlier years is less certain, as might be expected with the larger probable errors. The shape of the curve shows also that a single harmonic will not be sufficient to represent the observations, but that higher harmonics would have to be included. Hence it

seems best to give the results in two groups, the first, A, being the means from four periods of 14 lunations each to lunation 56, and the second, B, a similar mean to the end of the series.

Table IX contains the means for these two groups and also those for the whole series. The column "Mer." contains the means for all the meridian observations giving half-weight to those of Mösting A. The mean results for the separate series are given in the last three columns. The Greenwich series stops at the end of 1930, that from Washington at the end of 1931; the latter is not corrected for the difference between the 6- and 9-inch transits.

TABLE IX

Fourteen-month Oscillation in $-\delta\beta$. *Units of* 0″·01

	Occ.			Mer.	*GL*	*W*	*GA*
	A	B	A+B				
1	−61	−29	−46	− 2	− 4	− 5	+ 6
2	−25	−21	−23	+ 9	+10	+ 2	+23
3	−21	−20	−21	+21	+21	+16	+33
4	−27	−22	−25	+29	+33	+17	+14
5	−13	−28	−21	+20	+23	+11	+34
6	− 6	−29	−18	+23	+36	+ 4	+34
7	−15	− 6	+ 4	− 2	−23	+21	− 4
8	+50	+15	+32	−18	−29	− 6	−19
9	+57	+40	+48	+11	+24	− 5	+17
10	+45	+59	+51	−11	+ 3	−14	−34
11	+38	+44	+40	−28	−39	−11	−39
12	+ 9	+29	+19	−25	−37	−13	−26
13	− 3	− 6	− 5	−16	−16	−12	−23
14	−38	−29	−34	−11	− 9	−15	− 9

The similarity of the groups A, B shows that the period of the oscillation in the occultations cannot differ much from fourteen months. The combined meridian observations show a similar oscillation of somewhat smaller amplitude but *with the opposite phase.* The Washington observations give a smaller amplitude than any of the other series, and are remarkably smooth. There is possibly a smaller annual term in these series, but the material is not sufficient to give it in the presence of the fourteen-month term within the probable errors.

21. The annual oscillation in the longitude as found from the occultations has been discussed on several occasions by Dr. Brouwer and myself; Brouwer (*A.J.*, 946) found that in the years 1927–28 the bright stars (those listed in the almanacs for occultations) showed its presence, while it was absent from the occultations of the faint stars. The smaller annual term with a decidedly different phase in the Washington observations, H. R. Morgan (*loc. cit.*, No. 17) considered might be due to the changing aspect of the Moon's limb to the threads of the transit instrument combined with the systematic effect of the shape of the limb.

The investigations of Wood and Brouwer suggest a source in the star

catalogues, especially the Washington Zodiacal Catalogue, but the coefficient is decidedly larger than is to be expected from such a source, and it has tended rather to increase with the increased accuracy obtainable for the phases of the faint stars during 1930–31. The actual coefficient must also be larger than that found, owing to the classification by lunations rather than by values of the right ascension, if the source is to be found in the star places, since about one-third of the lunation is observed in the occultations selected for discussion.

Since the oscillation is fairly well represented by a single harmonic, it is probable that the annual means $\delta\lambda_l$ or $\delta\lambda_q$ are not much affected by its presence.

22. The fourteen-month oscillation in the latitude is much more puzzling. It cannot be due to the "variation of latitude," partly on account of its magnitude but mainly because the meridian observations show the term with the opposite sign. The most obvious suggestion for its presence is a libration effect combined with the systematic irregularities of the limb. In this case the principal apparent periods would be those of the differences between the synodic month and the nodical and anomalistic months. The former would be less than a year, and the latter rather less than fourteen months. We should naturally expect these oscillations to be present also in the longitude. This may be the fact, but the analysis by lunations is not sufficient to show them definitely. The main question that needs an answer, however, is why such a term should show its chief effect in the latitude. Owing to the smoothing effect of the analyses, the actual amplitude with this hypothesis must be considerably larger than that shown. With reference to the opposite phases given by the occultations and the meridian observations, the fact that the occultation results are determined wholly from the dark limb, while the meridian results arise wholly from observations of the bright limb, may assist in the explanation.

Summary of the Results

23. (*a*) The rate of rotation of the earth has been nearly constant in the years 1923–31. A small change appears to have occurred in 1927 and possibly in 1923–24. The probable error of the annual rate as found from all the observations does not appear to be greater than $0^s.2$.

(*b*) The annual means for the latitude deduced from the occultations show an oscillation with a period of nine years or longer. Part, if not all of this, is a distribution effect of the fourteen-month term and of insufficient material in 1923–26. There is little or no evidence of such an oscillation in the meridian observations.

(*c*) An oscillation with a period close to a year and an average coefficient near $0''.3$ exists in the longitude determined from the occultations, the coefficient increasing in the later years. A similar term exists in the meridian observations, but the amplitude is smaller and the phase different by at least 90°.

(*d*) Both occultations and meridian observations show an oscillation

in the latitude of about fourteen months period, the former with a half amplitude of about 0″·3 and the latter 0″·2. The phases are opposite, so that the whole amplitude of the difference between the two classes of observations with this period is about 1″.

(*e*) The suggested explanations of these oscillations require a fresh analysis of the material with the star positions corrected to a uniform system, and the systematic errors of the limb applied with the corresponding librations. A new direct determination of the latter is desirable.

Yale University:
1933 *May* 16.

THE VARIABLE SPECTRUM OF γ CASSIOPEIÆ.

(Second Paper.)

William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.).

In a previous communication * I published the evidence to show that the spectrum of γ Cassiopeiæ (H.D. 5394, 1900 R.A. $0^h\ 50^m{\cdot}7$, Dec. +60° 11′, Ptg. Mag. 2·01, Sp. Type Boemn7) was variable. The observations covered the period 1921 October 1 to 1933 February 14 and numbered 68 in all.

From the middle of 1928 (August 26) up to the last photograph discussed the observations were numerous (55 in number) and were well distributed over this period. Before that, however, *i.e.* from 1921 October 1 to 1927 October 10, they were rather sparse—only 13 in number—but included one or more photographs each year.

The whole series of observations not only showed that the difference of intensity between the bright components of each of the hydrogen lines definitely varied, but that this variation suggested a period of about four years. In this paper I referred to the deductions made by Sir Norman Lockyer † from the photographs secured at the Solar Physics Observatory at South Kensington between 1888 November 20 and 1894 November 16, and I was desirous of examining these negatives to see whether the variations noted in the Sidmouth photographs were apparent in these negatives. In the above reference Sir Norman Lockyer gave a complete list (p. 175) of the photographs which he employed in his discussion.

These negatives were all taken to the Solar Physics Observatory at Cambridge when the Kensington Observatory was transferred there, but owing to the kindness of the present Director, Professor F. J. M. Stratton, I have had the loan of some of them for examination, and I should like here to express to him my best thanks.

The Kensington photographs taken between 1880 November 20 and

* *M.N.*, **93**, 362, 1933.
† *Proc. R.S.*, **57**, 173, 1894.

1892 January 8, 27 in number, were all, with two exceptions, taken with a 6-inch prismatic camera of 90 inches focal length having a prism of 7°·5. These negatives, which I have not received, have probably so small a dispersion that they would not separate the bright components of the hydrogen lines. The two exceptions, also not sent, mentioned above, were taken with the 10-inch prismatic camera carrying two prisms of 7°·5 each.

The negatives received extend from 1892 May 10 to 1896 April 1, and form a good and valuable series. Two other negatives, taken in 1900 January 24 and 1901 December 3, were also sent, making 30 in number. These were all taken with the same 6-inch objective but carried a prism of 45° angle.

A further eight negatives, covering the period 1903 September 9 to 1904 September 19 have also been forwarded, and these were secured also with the 6-inch objective but carrying two prisms of 45° angle each. The first two of these negatives are too poor to discuss, and the last four were all taken in one month, so that leaves only three available months, namely, 1904 July to September, for estimate values. Reference to these values will be made later.

With regard to the main series of negatives here discussed, it may be mentioned that the dispersion by no means comes up to that employed at Sidmouth. The dispersion for the Sidmouth 9-inch prismatic camera of 120 inches focal length with its 45° prism is 46·0 mm. between $H\beta$ and $H\delta$, while that for the Kensington prismatic camera is 38·6 mm. for the same region.

Only two lines can be used for this inquiry, namely, $H\gamma$ and $H\delta$, because, chiefly owing to lack of sufficient exposure, the images of $H\beta$ and $H\epsilon$ are too faint for examination. In a few cases these lines are well visible, showing that it is not want of sensitiveness of the plates that is at fault.

Coming now to the discussion of the negatives, 25 only have been used. The two negatives taken in 1900 and 1901 are dealt with separately later on as they are too far away in time to be coupled up with the others, and three others were not sufficiently good.

The method of determining the differences of intensity of the bright components of the hydrogen lines by eye estimation was exactly the same as that employed in my previous paper, and the resulting estimates treated in the same way.

In the following table (Table I) the numbers of the negatives apply to the S.P.O. Cambridge numbers. The estimates are given for $H\gamma$ and $H\delta$ lines only. The headings (Q) of columns (5) and (7) denote the quality of the definition of the lines for estimation, and under these (M) means "moderate" and (P) "poor": a blank space denotes that the estimate is "good."

In making the estimates the negatives were examined in quite a miscellaneous order and two sets of estimates were made with a time interval of two weeks. The estimate values given in the table are the means of these two sets. At the end of the table the details of the two negatives taken in 1900 and 1901 and the six negatives taken in 1904 are given.

Table I

Neg. No.	Date	*Hγ* E	*Hγ* Q	*Hδ* E	*Hδ* Q	Neg. No.	Date	*Hγ* E	*Hγ* Q	*Hδ* E	*Hδ* Q
24	1892 June 29	−3·5		−5	M	306	1894 Nov. 16	0	M		
27	,, July 4	−2	M	−5	P	315	,, Dec. 12	0			
28	,, ,, 22	−8		−5	P	336	1895 Jan. 28	0		0	
53	,, Sept. 14	−2		−2		381	,, July 6	−1		+1	M
99	,, Oct. 18	−6·5		−5	M	389	,, Nov. 23	−2			
143	1893 Aug. 8	−5	M			390	,, Dec. 3	−1		−1	
145	,, ,, 16	−1		−1		404	1896 Apr. 1	−1		0	
192	,, ,, 22	+1	M	−1	P						
169	,, Sept. 4	−3·5		−5							
174	,, ,, 7	−3·5		−5		451	1900 Jan. 24	0		0	
217	,, Oct. 6	−1		−1		565	1901 Dec. 3	0		0	
203	,, ,, 26	0	P								
229	,, Dec. 14	0		−1		680	1904 July 6	0	P	−2	P
252	,, ,, 21	0		−1		681	,, Aug. 24	−2		0	M
236	1894 Jan. 12	−3·5		−3·5	M	684	,, Sept. 3	−2	M	−2	
237	,, ,, 18	−2		−5		687	,, ,, 7	−2		−5	
291	,, Oct. 16	0		−2		698	,, ,, 19	−2		−2	M
301	,, Nov. 11	+5		+2	P	695	,, ,, 20	0	P	−2	P

Fig. 1 represents a graphical representation of the main series of values given in Table I, but when more than one value is given for any particular month in that table the better quality estimate has been selected, or if they are of the same quality the mean has been employed. The scale for the original estimates is given on the left and the conversion into figures on the right. The vertical lines represent the month of January of each year. The full dots represent the "good" estimates, the circles the "moderate" and the circles with a question mark the "poor" ones, and each of these is plotted against the year and month in which it occurs.

The estimates for the bright components of the hydrogen lines *Hγ* and *Hδ* are plotted separately, and through each of them a smooth curve has been drawn to pass through the values as close as possible. These curves show a great similarity to each other, but there seems to be a small difference in phase.

It will be seen at once that the curves satisfy the observations much better than a horizontal straight line. If this be so, then the photographs indicate that the differences of intensity of the bright components of each of these hydrogen lines are not constant, and that therefore the star has a variable spectrum, thus corroborating the main result of my previous paper.

The next points to be considered are the epochs of maximum and minimum and the length in time of the variation. The mean maximum of the two curves falls about 1895 February. As the minima in my former communication, with the exception of that in 1933, do not go below the estimate $V > R$ or -5, the curves in the present series were drawn so as not to go below this value. Thus it will be seen that the minima here

indicated seem to be separated in each curve by four years and two months or by about four years.

In my previous paper I suggested that the Sidmouth observations indicated maxima in about March of the years 1931, 1927 and 1923, and a periodicity of approximately four years.

Calculating back the epoch of the maximum for this earlier date covered by the Kensington photographs, it should be found to take place in about 1895 March. This is very close to the date when it actually happened, namely, 1895 February, as indicated by the present curves.

It can be concluded therefore that these older observations of γ Cassiopeiæ

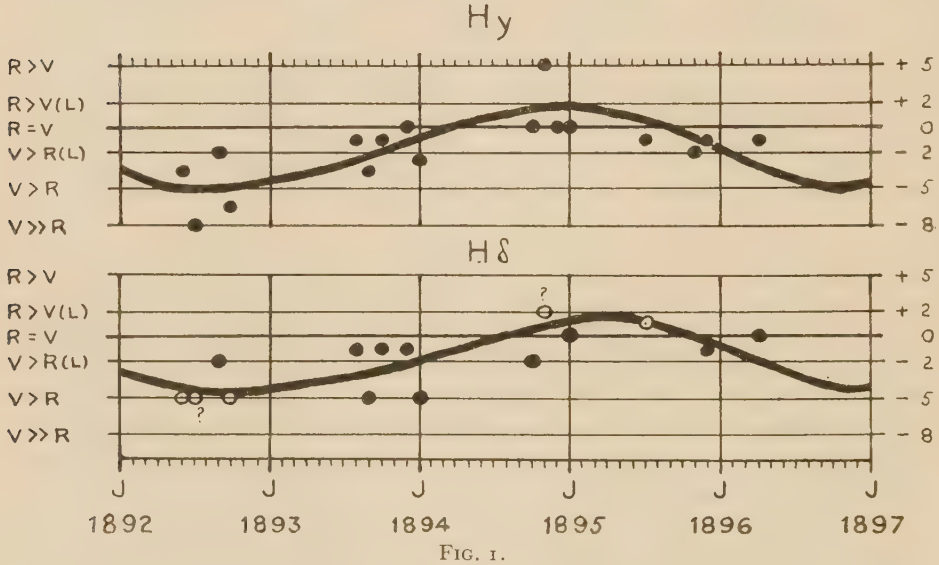

Fig. 1.

lend greater weight to the continuity of a four-year period as previously suggested.

Another point of particular interest in both these curves is that for the most part they both lie below the "zero" or "$V = R$" line. This peculiarity was specially noticeable in my previous curves, although attention was only drawn * to the fact that the wedge-measure curve was symmetrical about the "zero" line.

Thus this similarity of behaviour in both the quite independent series of observations points again to the sameness of type of the variation.

Reference may now be made to the other few negatives which cannot be coupled up with the main series. Two of these negatives were taken in 1900 January and 1901 December (see Table I) with the same instrument as the main series.

The estimated differences of intensity between the bright components of $H\gamma$ and $H\delta$ are zero in each case. The date of the epoch of maximum which immediately precedes them is 1899 March, so that the negatives were

* *M.N.*, 93, 370, 1933.

taken about 10 and 33 months after this maximum. Now about 1900 January the intensity-difference curves should cross the "zero" or "$R = V$" line on the downward side of the curve and should cross it again in about 1901 December on the upward trend of the curve. The above negatives therefore fall well into place with their expected positions.

With regard to the eight negatives taken with the 6-inch objective with two 45° prisms, the list being given at the bottom of Table I, two were not good enough for estimates to be made. The values for the others are given in Table I. Using only the "good" estimates, those for $H\gamma$ are -2 and -2 for August and September respectively, and for $H\delta$ $-3{\cdot}5$ for September. If 1903 March be the calculated date for the maximum preceding these dates, then the values for these two months should be between -2 and -5. These agree fairly satisfactorily with the actual estimated values.

It may be concluded therefore that the older Kensington photographs endorse in a satisfactory manner the Sidmouth deductions, not only in the lack of constancy of the spectrum of γ Cassiopeiæ, but in the probable four-year variation and the predominance of the intensity of the violet over the red components in the bright hydrogen lines.

In my last paper the observations discussed extended up to the month of February of the present year. It may be stated here that, since that communication was written, the values derived for the following months, namely, March, April and May, are still the same for all the hydrogen lines as they were in February. This indicates that the minima of the curves have been reached, as approximately shown in the figure (p. 369) of that paper.

In conclusion, it may be mentioned that I became aware of another series of photographs of the spectrum of γ Cassiopeiæ through Mr. Harold Thomson. This series was taken by Mr. Charles F. Butterworth of Port St. Mary, Isle of Man, who secured his photographs during the period 1916 January 24 to 1920 January 26, making 50 negatives in all. On request, Mr. Butterworth kindly sent me the negatives for examination, but unfortunately the dispersion was too small to bring out the details required. I should like to take this opportunity of thanking him for the loan of these valuable negatives.

Summary

1. In the March number of the *M.N.* (**93**, 372, 1933) it was shown that, from observations made at Sidmouth between 1921 and 1933, the spectrum of γ Cassiopeiæ was not constant but exhibited a variation of about four years in length.

2. An examination of photographs taken at the Solar Physics Observatory at South Kensington between 1892 June and 1896 April corroborates both the lack of constancy and the four-year period.

OBSERVATIONS OF V CASSIOPEIÆ (CH. 8324) IN 1926–33.

A. N. Brown, M.A.

The 311 observations of the long-period variable V Cassiopeiæ recorded below form a continuation to those published seven years ago in *M.N.*, **86**, 631–636, 1926. Light curves have been drawn from them, from which the following table has been compiled :—

Maxima			Minima		
Date	Julian Date	Mag.	Date	Julian Date	Mag.
1926 June 15	4682	8·0	1926 Oct. 21	4810	12·5
1927 Feb. 6	4918	7·6	1927 June 1	5033	12·3
1927 Sept. 21	5145	7·9	1928 Jan. 17	5263	12·3
1928 May 14	5381	8·2	1928 Aug. 25	5484	12·6
1928 Dec. 13	5594	7·7	1929 Apr. 3	5705	12·2
1929 July 23	5816	8·2	1929 Nov. 23	5939	12·4
1930 Feb. 23	6031	8·1	1930 July 5	6163	12·0
1930 Oct. 17	6267	7·8	1931 Feb. 14	6387	11·3
1931 May 25	6487	8·0	1931 Sept. 21	6606	12·2±
1932 Jan. 17	6724	7·9	1932 May	...	...
1932 Sept. 2	6953	8·0	1932 Dec.	...	...
1933 Apr. 20	7183	8·0			

It is unfortunate that owing to persistently bad observing weather no dates can be given for the minima which occurred in 1932 May and December.

In the note appended to the observations published in 1926 it was mentioned that the period had been gradually lengthening up to that time, it being 231·7 days for the years 1921–26. During the years covered by the observations now under consideration this process appears to have ceased, the mean period now being 225·9 days. The mean period of the variable since my observations began in 1909 is now 227·6 days. Minima as well as maxima have been used in deducing these periods.

Usually the rise to maximum, for which the average time from 1926 to 1933 is 106·4 days, is more rapid than the fall to minimum (118·8 days), but, as has been noted previously, this is not invariably the case.

My observations of this variable from 1909 to 1929 were fully discussed by Miss Mary A. Blagg in *M.N.*, **90**, 552–560, 1930. The same telescopes have been used as previously, namely, 5-inch and 3-inch refractors with powers varying from 30 to 150.

The Observations

Date	Julian Date	Comparisons	Ded. Mag.
1926			
May 20	4656	5(2)*v*(1)7	8·5
31	4667	4 − 4, 5 + 2	8·1
June 5	4672	4 − 3, 5 + 3	8·0
15	4682	4(1)*v*(1)5, 4 − 3	8·0
22	4689	4(1)*v*(1)5	8·0
26	4693	4 − 3, 5 + 3	8·0
July 13	4710	5(1)*v*(1)7	8·5
15	4712	5(3)*v*(1)7	8·5
21	4718	7(1)*v*(1)11	8·9
30	4727	11(2)*v*(1)15	9·4
Aug. 4	4732	15(1)*v*(1)17	9·6
7	4735	17(2)*v*(1)21, 22 + 1, 24 + 3	9·7
14	4742	22(3)*v*(1)32	10·2
22	4750	32 − 2, = 34	10·5
26	4754	34(1)*v*(1)43	10·7
Sept. 4	4763	43(3)*v*(1)51	11·1
10	4769	51(2)*v*(1)55	11·3
13	4772	55(1)*v*(2)64	11·5
19	4778	64(1)*v*(1)69	11·9
26	4785	= 69	12·0
30	4789	69(1)*v*(2)73	12·1
Oct. 2	4791	69(2)*v*(1)73	12·2
10	4799	= 73, 75 + 1	12·3
18	4807	75 + 1	12·3
26	4815	73 − 1, 75 − 1	12·5
30	4819	= 73, 75 − 1	12·4
Nov. 5	4825	73 + 1, 75 + 1	12·3
7	4827	69(3)*v*(1)73, 75 + 2	12·2
12	4832	64(4)*v*(1)69	12·0
Dec. 2	4852	32(2)*v*(1)43	10·7
6	4856	22(3)*v*(1)32	10·2
8	4858	22(3)*v*(1)32, 34 + 2	10·2
15	4865	17(2)*v*(1)22	9·8
27	4877	14(1)*v*(1)15	9·5
29	4879	9(2)*v*(1)14	9·2
1927			
Jan. 4	4885	7(1)*v*(4)11	8·7
16	4897	4(5)*v*(1)5	8·2
21	4902	4(1)*v*(1)5	8·0
25	4906	4 − 2, 5 + 4	7·9
27	4908	4 − 2, 5 + 4	7·9
Feb. 2	4914	4 + 1	7·6
4	4916	4 + 1	7·6
24	4936	4 − 4, 5 + 2	8·1
Mar. 7	4947	5 − 2, 7 + 1	8·5
23	4963	= 17	9·6
27	4967	22(2)*v*(1)34, 32 + 2	10·2
30	4970	22(3)*v*(1)32	10·2

Date	Julian Date	Comparisons	Ded. Mag.
1927			
Apr. 7	4978	34(2)*v*(1)43	10·7
19	4990	51 + 1	11·1
26	4997	55 − 3, = 64	11·8
May 3	5004	64(2)*v*(1)69	11·9
8	5009	= 69	12·0
11	5012	69 − 1	12·1
June 3	5035	69(3)*v*(1)73, 75 + 1	12·3
21	5053	64(3)*v*(1)69	12·0
July 5	5067	32(4)*v*(1)43	10·8
26	5088	22(2)*v*(1)32	10·2
Aug. 2	5095	15(2)*v*(1)17	9·6
11	5104	7(1)*v*(1)11	8·9
16	5109	7(3)*v*(2)11	8·9
23	5116	4(5)*v*(1)5	8·2
Sept. 1	5125	4(2)*v*(1)5, 5 + 2	8·1
7	5131	4(1)*v*(1)5	8·0
22	5146	4(1)*v*(2)5	7·9
25	5149	4 − 2, 5 + 4	7·9
29	5153	4 − 3, 5 + 3	8·0
Oct. 2	5156	4(1)*v*(1)5	8·0
6	5160	4(1)*v*(1)5	8·0
9	5163	4(2)*v*(1)5	8·1
17	5171	7(2)*v*(3)11	8·8
23	5177	9(3)*v*(1)14	9·3
31	5185	15(1)*v*(2)22	9·6
Nov. 9	5194	32 + 1, 34 + 2	10·2
16	5201	34(3)*v*(2)43, 40 + 2	10·6
24	5209	43(2)*v*(1)51	11·1
1928			
Jan. 9	5255	69 − 3, 75 + 1	12·3
11	5257	75 + 1	12·3
16	5262	= 73	12·3
22	5268	73 + 0·5, 75 + 1	12·3
27	5273	75 + 1	12·3
Feb. 13	5290	64(2)*v*(1)69	11·9
19	5296	= 64	11·8
22	5299	= 55	11·4
25	5302	43(3)*v*(1)51	11·1
Mar. 3	5309	32 + 1, 34 + 2	10·2
10	5316	22(2)*v*(1)34, 32 + 1	10·2
13	5319	22 + 1	9·8
30	5336	9(1)*v*(1)14	9·2
Apr. 4	5341	7(3)*v*(1)11	9·0
18	5355	5 − 2, 7 + 1	8·5
28	5365	5(2)*v*(1)7	8·5
May 5	5372	5(1)*v*(2)7	8·4
14	5381	5 + 1	8·2
18	5385	4(5)*v*(1)5	8·2
25	5392	5(2)*v*(1)7	8·5

Date	Julian Date	Comparisons	Ded. Mag.
1928			
June 2	5400	=7	8·6
10	5408	11(3)v(1)15, =14	9·4
16	5414	15(2)v(1)22	9·8
22	5420	22(4)v(1)32, 34+2	10·2
July 14	5442	=64	11·8
17	5445	64(1)v(2)69	11·9
21	5449	64(3)v(1)69	12·0
28	5456	69(3)v(1)73	12·2
Aug. 8	5467	73+1, 75+1	12·3
18	5477	75−1	12·5
28	5487	75−2	12·6
Sept. 4	5494	=75	12·4
9	5499	69(3)v(1)73, 75+2	12·2
17	5507	69(3)v(2)73, 75+3	12·1
20	5510	64(3)v(1)69	12·0
Oct. 1	5521	43(2)v(1)51	11·1
4	5524	34(2)v(1)43	10·7
11	5531	32(1)v(3)34	10·3
17	5537	22(1)v(2)32	10·0
25	5545	=14, 15+1	9·4
Nov. 7	5558	9(3)v(1)15, =14	9·4
9	5560	9(3)v(1)14, 15+1	9·3
17	5568	7(2)v(1)11	8·9
25	5576	=5	8·3
Dec. 1	5582	4(4)v(1)5	8·2
5	5586	4(1)v(1)5	8·0
7	5588	4−2, 5+4	7·9
17	5598	=4	7·7
20	5601	4−2, 5+4	7·9
26	5607	4−4, 5+2	8·1
1929			
Jan. 12	5624	=5	8·3
15	5627	7(1)v(2)11	8·8
19	5631	9(2)v(1)11, 11+1	9·0
26	5638	15−2, =17	9·7
Feb. 3	5646	32(1)v(2)43, 34−1	10·5
12	5655	32(4)v(1)43, 34−3	10·7
28	5671	51(1)v(2)55	11·3
Mar. 2	5673	51(1)v(2)55	11·3
6	5677	=64	11·8
10	5681	64(1)v(3)69	11·9
20	5691	<64	<11·8
Apr. 1	5703	69(2)v(1)73, 75+3	12·2
5	5707	69(3)v(1)73	12·2
18	5720	<64	<11·8
May 2	5734	64(1)v(2)69	11·9
9	5741	51+1	11·1
12	5744	43(2)v(1)51	11·1
15	5747	32(3)v(1)43, 34−3	10·7
19	5751	32(2)v(1)43, 34−1	10·6
25	5757	32(1)v(3)43, =34	10·4

Date	Julian Date	Comparisons	Ded. Mag.
1929			
June 10	5773	9(2)v(1)14, 15+3	9·2
18	5781	7(1)v(4)11, 6−2	8·8
24	5787	5(2)v(1)7	8·5
July 8	5801	4(3)v(2)5	8·1
11	5804	4−3, 5+2	8·1
14	5807	4(3)v(1)5	8·2
17	5810	4(4)v(1)5	8·2
29	5822	4−4, 5+1	8·2
Aug. 7	5831	4−5, 5+1	8·2
13	5837	=5	8·3
25	5849	7(2)v(1)11	8·9
Sept. 1	5856	7(3)v(1)11	9·0
3	5858	11(2)v(1)15, 14+1	9·3
6	5861	14−1, 15+1	9·5
11	5866	15(2)v(1)17	9·6
23	5878	22(2)v(1)32	10·2
26	5881	22(3)v(1)34, 32+1	10·2
Oct. 9	5894	32(4)v(1)43	10·8
16	5901	51(2)v(1)55, 51−2	11·4
26	5911	64(1)v(2)69	11·9
Nov. 1	5917	64(3)v(1)69	12·0
3	5919	69+1	11·9
10	5926	69(3)v(1)73	12·2
12	5928	=75	12·4
28	5944	73−1, =75	12·4
Dec. 2	5948	75+1	12·3
18	5964	51(2)v(1)64	11·6
26	5972	43(2)v(1)51	11·1
30	5976	43(3)v(2)51	11·1
1930			
Jan. 5	5982	32−1, =34	10·4
19	5996	15(4)v(1)22	9·8
25	6002	9(3)v(1)14	9·3
Feb. 5	6013	4(5)v(1)5	8·2
9	6017	4(5)v(1)5	8·2
16	6024	4−4, 5+2	8·1
20	6028	5+2	8·1
26	6034	4(1)v(1)5	8·0
Mar. 1	6037	4(3)v(2)5	8·1
7	6043	4(2)v(1)5	8·1
18	6054	4(5)v(1)5	8·2
21	6057	4−5, 5+1	8·2
29	6065	5(2)v(1)7	8·5
Apr. 2	6069	=7	8·6
22	6089	11(1)v(1)15	9·3
May 18	6115	32(2)v(3)43, 34−2	10·6
June 5	6133	51(1)v(1)55	11·3
28	6156	64(3)v(1)69	12·0
July 19	6177	64(2)v(1)69	11·9
Aug. 16	6205	15−2, =17, 19+1	9·6
19	6208	14−3, 15(1)v(1)22	9·7

Date	Julian Date	Comparisons	Ded. Mag.
1930			
Aug. 25	6214	11(1)v(2)15, 14 +2	9·2
Sept. 1	6221	7(3)v(1)11	9·0
15	6235	7(1)v(4)11	8·7
24	6244	4(4)v(1)5	8·2
Oct. 9	6259	4 −2, 5 +4	7·9
12	6262	4(1)v(4)5	7·8
19	6269	4 −1, 5 +5	7·8
24	6274	4(1)v(5)5	7·8
Nov. 11	6292	4(2)v(1)5	8·1
13	6294	4 −4, 5 +2	8·1
16	6297	=5, 7 +2	8·4
21	6302	7(1)v(2)11	8·8
Dec. 13	6324	22(1)v(1)34	10·2
21	6332	22(2)v(1)34, 32 +2	10·2
25	6336	32 +1, 34 +2	10·2
1931			
Jan. 6	6348	32(3)v(1)43, 34 −2	10·7
20	6362	51 −1	11·3
24	6366	51 +1	11·1
26	6368	51(1)v(1)55	11·3
Feb. 7	6380	51(1)v(2)55	11·3
10	6383	51 −1	11·3
12	6385	51(1)v(1)55, 64 +5	11·3
21	6394	51 −1, 55 +1	11·3
Mar. 8	6409	51 −1	11·3
14	6415	43(2)v(1)51	11·1
Apr. 11	6443	7(1)v(2)9	8·7
17	6449	=7	8·6
27	6459	5 +1	8·2
May 5	6467	4(2)v(1)5	8·1
28	6490	4(1)v(1)5	8·0
June 12	6505	4(4)v(1)5	8·2
17	6510	5 +1, 7 +3	8·3
21	6514	4(5)v(1)5	8·2
26	6519	5(2)v(1)7	8·5
July 16	6539	9(2)v(1)14	9·2
Aug. 17	6571	43(1)v(1)51	11·1
Sept. 6	6591	=69	12·0
8	6593	64(3)v(1)69	12·0
15	6600	69(2)v(1)73	12·2
Oct. 11	6626	=51	11·2
20	6635	43(1)v(1)51	11·1
25	6640	43 +1	10·8
Nov. 1	6647	32(2)v(1)43, 34 −2	10·7
4	6650	32 −1, =34	10·4
9	6655	=32	10·3
13	6659	22(4)v(1)32	10·2
Dec. 7	6683	5(2)v(1)7	8·5
17	6693	5 +1	8·2
30	6706	4 −4, 5 +2	8·1

Date	Julian Date	Comparisons	Ded. Mag.
1932			
Jan. 8	6715	4 −2, 5 +4	7·9
11	6718	4 −3, 5 +3	8·0
Feb. 3	6741	4 −4, 5 +2	8·1
Mar. 1	6768	15(2)v(1)22	9·8
3	6770	17(1)v(1)22	9·8
6	6773	17(3)v(2)22	9·8
9	6776	15(4)v(1)22, 22 +1	9·8
12	6779	22(3)v(1)32, 34 +2·5	10·2
18	6785	32 −1, =34	10·4
23	6790	34(3)v(1)51	11·0
Apr. 8	6806	51(2)v(1)55	11·3
11	6809	64 +2	11·6
22	6820	64 −2	12·0
May 17	6845	< 64	< 11·8
June 8	6867	51(1)v(1)55	11·3
14	6873	51(2)v(1)55	11·3
26	6885	15 −2	9·7
July 2	6891	9(1)v(1)14	9·2
9	6898	7(3)v(1)11	9·0
25	6914	7(3)v(1)11	9·0
27	6916	6(3)v(1)12, 7(3)v(1)11	9·0
31	6920	=7	8·6
Aug. 4	6924	=7	8·6
7	6927	5(1)v(1)7	8·5
12	6932	5(1)v(1)7	8·5
19	6939	4(3)v(1)5	8·2
26	6946	4(1)v(1)5	8·0
30	6950	4(1)v(1)5	8·0
Sept. 3	6954	4(1)v(1)5	8·0
9	6960	4 −3, 5 +3	8·0
26	6977	9(1)v(1)14	9·2
28	6979	9(3)v(1)14, 15 +2	9·3
Oct. 1	6982	15(2)v(1)17	9·6
6	6987	17(2)v(1)22	9·8
9	6990	22(1)v(2)34	10·1
18	6999	32(1)v(1)34	10·4
26	7007	51 +2	11·0
31	7012	51 +1	11·1
Nov. 5	7017	=51	11·2
7	7019	51 −2, =55	11·4
21	7033	64(2)v(1)69	11·9
28	7040	64(1)v(1)69	11·9
Dec. 1	7043	=73, 75 +1	12·3
30	7072	=73, 75 +1	12·3
1933			
Jan. 12	7085	< 64	< 11·8
21	7094	=55	11·4
23	7096	51 −1	11·3
25	7098	=51	11·2
30	7103	=43	10·9
Feb. 11	7115	22(2)v(1)32, 34 +2	10·2
13	7117	22(2)v(1)32	10·2

Date	Julian Date	Comparisons	Ded. Mag.
1933			
Feb. 17	7121	=22	9·9
23	7127	15(2)*v*(1)22	9·8
28	7132	15(1)*v*(2)22	9·6
Mar. 7	7139	=14, 15 +1	9·4
13	7145	7(2)*v*(1)11	8·9
20	7152	7(1)*v*(3)11	8·7
25	7157	5(1)*v*(1)7	8·5
27	7159	5(1)*v*(2)7	8·4
1933			
Mar. 30	7162	4(3)*v*(1)5	8·2
Apr. 3	7166	4 −4, 5 +2	8·1
13	7176	4(1)*v*(1)5	8·0
19	7182	4 −3, 5 +3	8·0
30	7193	4 −3, 5 +3	8·0
May 5	7198	5(3)*v*(1)7	8·5
14	7207	7(3)*v*(1)11	9·0
22	7215	15(2)*v*(1)17	9·6

The Comparison Stars Used

Hagen's Number	H.P. Magnitude	Hagen's Number	H.P. Magnitude	Hagen's Number	H.P. Magnitude
4	7·7	17	9·6	43	10·9
5	8·3	19	9·7	51	11·2
6	8·6	21	9·8	55	11·4
7	8·6	22	9·9	64	11·8
9	8·9	24	9·9	69	12·0
11	9·1	32	10·3	73	12·3
14	9·4	34	10·4	75	12·4
15	9·5	40	10·7		

The values given above are obtained from the H.P. column in Hagen's *Atlas Stellarum Variabilium*, Series VI.

Brackenhurst,
Bucklebury Common, Berks:
1933 *June* 1.

ON THE EXPANDING UNIVERSE AND THE TIME-SCALE.

W. de Sitter.

1. I may be allowed to begin by recalling briefly the principal facts relating to the theory of the expanding universe.

The line-element of time-space of a universe containing matter and radiation, homogeneously and isotropically distributed over the three-dimensional space, is

$$ds^2 = R_1^2[-y^2 d\sigma^2 + d\tau^2], \qquad (1)$$

where $d\sigma$ is the line-element of a three-dimensional space of constant unit curvature k ($k = -1$, 0 or $+1$) and y is a function of τ alone. R_1 is a constant, which arises in the integration of the equation of energy.* In the simple

* Empty universes, having $R_1 = 0$, are not considered in this paper. The actual universe is not empty.

case of no pressure, which is a good approximation to the truth, the integral of this equation is

$$\kappa\rho R_1^2 y^3 = 3, \tag{2}$$

where ρ is the proper density. The factor 3 arises naturally: it is the number of dimensions of the three-dimensional space. In the actual universe, containing material pressure and radiation, and in which the distribution of matter and radiation differs from complete homogeneity and isotropy, the integral is more complicated, and cannot be explicitly stated, but it still exists, and contains a constant of integration. Since the integral must in the idealized case degenerate into (2), this constant is the same R_1. It depends on the material contents of the universe, and is the natural unit in which lengths and times are measured. In the actual universe it is probably of the order of 10^{27} cm. or 10^9 years. The instantaneous curvature of three-dimensional space is k/R^2, where $R = R_1 y$.

The differential equation determining y as a function of τ, *i.e.* Lemaître's equation, is, in the simple case in which the pressure is neglected,

$$\left(\frac{dy}{d\tau}\right)^2 = \frac{1}{y} - k + \gamma y^2, \tag{3}$$

where $k = -1$, 0 or $+1$ as before, and $\gamma = \frac{1}{3}R_1^2\lambda$, λ being the "cosmical constant" introduced by Einstein. The value of λ, or γ, is entirely unknown.

The acceleration, we may remark in parenthesis, is

$$\frac{d^2y}{d\tau^2} = \gamma y - \frac{1}{2y^2}. \tag{3'}$$

It consequently is independent of the curvature k, and can only be positive for positive values of γ.

The possible solutions of (3) are of three types, which I have called the expanding universes of the first and of the second kind and the oscillating universes, depending on the values of k and γ. For negative values of γ the solution is of the oscillating type, the value of y oscillating between zero and a maximum in a finite period (of the order of π). For $\gamma > +4/27$ the solution is an expanding universe of the first kind, in which y increases from zero to infinity, the velocity of increase $dy/d\tau$ being infinite both for $y = 0$ and for $y = \infty$. The time needed to increase from zero to a moderate value y is of the order of y itself. For very large y this time becomes of the order of log y, the solution approaching asymptotically to $y = e^{(\tau - \tau_0)\sqrt{\gamma}}$.* For values of γ between zero and $+4/27$ the solution is an expanding one of the first kind if k is either -1 or 0. For $k = +1$ two solutions are possible

* For large y the solution is:

$$\tau - \tau_0 = \frac{1}{\sqrt{\gamma}} lgy + \frac{k}{4\gamma^{3/2}} \cdot \frac{1}{y^2} - \frac{1}{6\gamma^{3/2}} \cdot \frac{1}{y^3} + \frac{3k^2}{32\gamma^{5/2}} \cdot \frac{1}{y^4} + \frac{3k}{20\gamma^{5/2}} \cdot \frac{1}{y^5} + \ldots$$

For small y it is:

$$\tau - \tau_0 = \tfrac{2}{3}y^{\frac{3}{2}}[1 + \tfrac{3}{10}ky + \tfrac{9}{56}k^2y^2 + (\tfrac{5}{48}k - \tfrac{1}{8}\gamma)y^3 + \ldots].$$

for these values of γ, viz. either an oscillating one with a period exceeding π and a maximum value of y smaller than 1·5, or an expanding universe of the second kind, in which y has a minimum value exceeding 1·5, and increases to infinity on both sides of the minimum. The time from the minimum to any moderate value of y is again of the order of y itself, and for very large y of the order of log y. In the limiting case $\gamma = 4/27$ the maximum value of y in the oscillating universe is 1·5 and the period becomes infinite, the maximum being reached asymptotically for $\tau = +\infty$, whilst in the expanding universe of the second kind the minimum is 1·5 and is reached asymptotically at the time $\tau = -\infty$. The particular solution $y = \text{constant} = 1{\cdot}5$ ($k = +1$, $\gamma = 4/27$) is "Einstein's universe," and is unstable. For $\gamma = 0$ the universe is expanding of the first kind for $k = -1$ or 0, and oscillating for $k = +1$.

Astronomical observations give us no means whatever to decide which of these possible solutions corresponds to the actual universe. The choice must, as Sir Arthur Eddington says, depend on æsthetical considerations. Reasons will be given below for preferring the minimum value $y = 0$ to a finite minimum, thus limiting the choice to the expanding universes of the first kind and the oscillating ones. Personally I have, like Eddington, a strong dislike to a periodic universe, but that is a purely personal idiosyncrasy, not based on any physical or astronomical data. The values of k and γ remain undetermined (except that negative values of γ are excluded if oscillating universes are not admitted), but the following considerations are independent of these values.

2. The short time-scale of these expanding universes has been repeatedly commented upon. With the exception of the very special case $k = +1$, $\gamma = 4/27$, the period in the oscillating case, and the time elapsed since y had its minimum value (either zero or a finite value) in the expanding case, is only a few thousand million years. The time-scale is short not only in the past, but also in the future; except in the special case $k = 0$, $\gamma = 0$, the value of y in the expanding universes of both kinds will be so large (say a thousand times its present value) that the universe will practically be disintegrated into independent galaxies, after a time of the order of 10^{10} years. The universe consists of galaxies, which are at present separated from each other by distances of the order of 30 or 50 times their diameters. The galaxies consist of stars, for which this ratio is of the order of 10^7 or 10^8, and the stars consist of electrons and protons, for which the same ratio is of the order of 10^5 (so far as the "diameter" of an electron or a proton has any meaning). The present structure of the universe, so different from that of the galaxies and of the stars in the ratio of the mutual distances and the diameters of their constituents, is, however, only an episode of a very ephemeral character, lasting not longer than a few times 10^{10} years. The constitution and the dimensions of the stars and the galaxies are not affected by the expansion of the universe.

Nevertheless the "age of the universe," *i.e.* the time since y passed through its minimum, has very generally been assumed to be also the age of the stars, and consequently the opinion has generally been held that the long time-scale of 10^{12} or 10^{13} years, demanded by modern theories for the evolution of the

stars, would have to be given up, and the theories of evolution would have to be modified so as to give a shorter time-scale. I think this identification of the time of minimum of y with the "beginning of the world" is entirely gratuitous. In the case of the expanding universes of the second kind it is at once evident that the minimum is not a very remarkable point on the curve at all. There is no singularity at that point and no discontinuity in the motion, no more than at the perihelion of a planetary or cometary orbit.

In the case of the expanding universes of the first kind and the oscillating universes, where the minimum of y is zero, the case is different. The equation (1) is an idealization of the problem, taking account of inertia only and neglecting gravitation, or, in other words, replacing the mutual gravitational action between the separate galaxies by the averaged action of all galaxies in the universe combined. This approximation is, of course, entirely sufficient so long as the mutual distances are large, but ceases to be an approximation when these distances become very small. The co-ordinates of a galaxy, expressed in natural measure, are $x_i = R_1 y \xi_i$, if ξ_i are the co-ordinates in the three-dimensional unit space of line-element $d\sigma$. Described in these co-ordinates x_i, the orbit of a galaxy in the idealized case, corresponding to the line-element (1), is a curve which near the origin assumes a parabolical form, having its apex in the neighbourhood of (but not at) the origin. The body passes through the origin at the time $t = t_0$, which is the same for all galaxies, being the time at which $y = 0$, and the velocity in natural measure at that time is the velocity of light. The accelerations are, however, different for the individual bodies, and at a small distance from the origin the velocities are also very different. The mutual perturbations near the origin will be large and different for different bodies, and the exact simultaneity and exact equality of all velocities will consequently be destroyed. An attempt has been made to estimate the amount of these perturbations, which will be referred to later on.

The conception of a universe shrinking to a mathematical point at one particular moment of time $t = t_0$ must thus be replaced by that of a near approach of all galaxies during a short interval of time near $t = t_0$. The minimum distance and the time when this is reached is different for each pair of galaxies, and the velocities, though very large, are not rigorously equal to the velocity of light, and are also individually different. The dimensions of the galaxies themselves are not directly affected. It should be remembered that the distribution of the stars in the galaxies is so very rare that they can easily penetrate each other. If we put a million galaxies in the space now occupied by one, the ratio of the mutual distances and the diameters of the stars will still be of the order of 10^5 or 10^6, and, though collisions or near approaches will be more frequent than at present, they are still exceptions, and there is no reason to suppose that the epoch of minimum y is of any special importance in the evolution of the stars.

I wish to emphasize that the above is not a new theory of the expanding universe, like Dr. Milne's, but simply an elaboration, or consequence, of Lemaître's general theory. The large velocities of the galaxies near $t = t_0$,

just like their present systematic positive radial motions, are not constants of integration, different for each individual body, but are imparted to them by the gravitational action of the whole of the universe. The individual peculiar velocities are assumed to be small and accidental.

3. The adoption of the long time-scale for the evolution of the stars dates from the publication of Eddington's mass-luminosity curve in 1924 March. Theories of the generation of energy in the stars were then developed to account for this long time-scale. Lately other theories, ascribing the generation of energy to atomic processes instead of to the annihilation of matter, have been proposed, leading to a time-scale of the order of 10^{10} years. Primarily, however, what led to the adoption of the long time-scale was not the belief in annihilation of matter or the corresponding theory of evolution, but the observational fact of the existence of the mass-luminosity relation. If the stars of different luminosity (and consequently, according to the spectrum-luminosity relation, of different spectrum) represent different stages in one and the same process of evolution, then the rate of this evolution is determined by the rate at which the star can lose mass, since it must necessarily slide down the mass-luminosity curve. As there seems to be no other method for a star of losing mass than by radiation, this rate is necessarily slow, and the time-scale is long. With a short time-scale the stars must retain practically the same luminosity during their whole life, and the "main series" does not represent a sequence of evolutionary stages, but a variety of individually different creations. Apart from any judgment regarding the relative merits of different theories of the generation of energy in a star, I think that the empirically ascertained mass-luminosity relation is a very strong argument in favour of the long time-scale. We must therefore accept the paradox that the stars are older than the universe, if by the "age of the universe" we mean the time elapsed since y passed through its minimum. It has been shown, however, that this minimum must not be conceived as the "beginning of the world," but as a transitory episode in the history of the universe, so that there is nothing paradoxical left in the paradox.

4. The time $t = t_0$ of the minimum of y is between 10^9 and 10^{10} years ago. Now it is worthy of remark that there are at least two other entirely independent lines of reasoning, which point to this same time, say 5.10^9 years ago, as a critical epoch.

The age of the Earth's crust is of the order of a few thousand million years. This determination, resting on the chemical analysis of minerals and the laws of radioactive processes, leaves only a small margin of uncertainty. The age of the Earth itself is probably not more, and possibly much less, than twice the age of the crust. All modern theories of the origin of the solar system agree in ascribing this origin to a near approach or a collision of the Sun with another star, consequently the age of the Earth is also the age of the solar system. Near approaches in our galactic system as it is now are extremely rare. Unless we are prepared to consider our own system as a freak, practically unique amongst stars, we must believe that about 5.10^9 years ago the chance of encounters must have been much greater than it is now. The near approach and mutual penetration of many galaxies

at that epoch, as explained above, does provide this increased chance of encounters.

The structure of our own galactic system, and of those external galaxies which are near enough to enable us to study their structure, is extremely complicated. The distribution of density is very irregular, and very far from homogeneous. The systems are all rotating in periods, which are of the order of a few hundred million years. As a consequence of this rotation the unhomogeneity of structure cannot exist for ever, but must be gradually smoothed out. The age of the galactic systems in their present form can therefore not be more than a small number of periods of revolution. Probably ten can still be considered as a small number in this respect, but almost certainly a hundred would be a large number. The galactic system and the spiral nebulæ must thus have acquired their present structure at a definite epoch in the past, their ages being of the order of $10 \times$ (a few times 10^8) years, *i.e.* of the same order of 5.10^9 years as the age of the planetary system and of the universe. It need not be pointed out expressly that again the near approach of all galaxies at that epoch affords sufficient explanation for the setting up of the rotation, the formation of the spiral arms and the unhomogeneous distribution of matter.

The great increase of density and consequent increased chance of encounters needed for the explanation of the origin of the spiral galaxies and the planetary systems is, however, only provided at the epoch of minimum if the universe is either an oscillating one or an expanding one of the first kind. In the expanding universes of the second kind the increase of density is hardly sufficient to produce these effects, since at the minimum the distances between the galaxies were probably still equal to several times the diameters.

5. The line-element (1) supposes a uniform density. If we wish to take account of the mutual gravitation of the individual galaxies, this must be replaced by a discontinuous distribution, the matter being concentrated in concrete separate points. The formation of a line-element corresponding to this discontinuous distribution is a formidable mathematical problem, which I have not attempted to tackle. In order to arrive at an estimate of the size of the perturbations in the motion of the galaxies by their mutual gravitational attraction, I have contented myself with considering only two galaxies, one of which is situated in the origin of co-ordinates. The problem thus reduces to that of finding the motion of a material point, expressed in natural measure, in the field produced by one galaxy at the origin of co-ordinates and the rest of the universe. The assumption is made that the line-element representing this field is of the form

$$ds^2 = -e^{-2\gamma}y^2 d\sigma^2 + e^{2\gamma}d\tau^2, \tag{4}$$

where R_1 has been taken as unit of length and time, and γ is supposed to be a function of the radius vector

$$r = y\chi$$

alone, χ being the radius vector in the unit space $d\sigma$. This line-element satisfies the field equations to the first order of γ, and it agrees, also to the

first order of γ, with that recently derived by Dr. McVittie,* if we take for γ the ordinary Newtonian potential :

$$\gamma = -\frac{m}{r}. \tag{5}$$

For distances r smaller than the diameter of a galaxy, *i.e.* when the two galaxies considered interpenetrate each other, (5) may, in analogy with Newtonian mechanics, be replaced by

$$\gamma = ar^2 + b, \tag{5'}$$

the constants a and b being so chosen that at the boundary, $(r = r_1)$, γ and $\gamma' = d\gamma/dr$ are the same by both formulæ. For m I have taken $\frac{1}{4}\cdot 10^{-10}$, which in the units chosen corresponds to about $10^{11}\odot$ for the sum of the masses. The "unperturbed" motion is a geodesic in the space $d\sigma$ described with a variable velocity. Expressed in the co-ordinates $x_i = y\xi_i$ it becomes a parabola-like curve, passing through the origin, the velocity at the origin being the velocity of light.† The "perturbation" is the difference between the geodesic corresponding to the line-element (4) and that corresponding to (1). The detailed computation of the perturbations will be published in one of the next numbers of the *B.A.N.* The perturbations are of the order of one or two parsecs, or a few years, both in the co-ordinates x_i and in the time τ_0 of passing through the origin. This rather small amount is the perturbation of the orbit of a galaxy by *one* other. That it is so small is evidently due to the enormous relative velocity, making the time during which there is any appreciable action very short. If there are n other galaxies their effects will be cumulative, and as a very rough approximation we can take their sum to be of the order of $\sqrt{n}$ times that by one unit. For $n = 10^6$ this would already give a total perturbation comparable with the dimensions of a galaxy. The total number of galaxies in the universe is, of course, much larger than 10^6, probably more nearly 10^{12}, if not infinite, but for such large numbers the proportionality of the total perturbation with $\sqrt{n}$ can no longer be supposed to be even a rough approximation.

Although this computation does not have the pretension to be more than a rough guess at the order of magnitude of the deviation of the motions in the actual universe from the idealized mathematical abstraction represented by (1), it seems to show that the hypothesis as to what will actually happen (or has happened) at the time $t = t_0$ developed in the paragraphs 2 and 4 is at least not improbable.

* *M.N.*, **93**, 325, 1933. McVittie's r is my χ, my r being his r_1. McVittie's $e^{\beta(t)}$ is my y.

† See *B.A.N.*, Nos. 193 and 200 (1930), and my Hitchcock Lectures, *University of California Publications in Mathematics*, **2**, No. 8, p. 186 (1933).

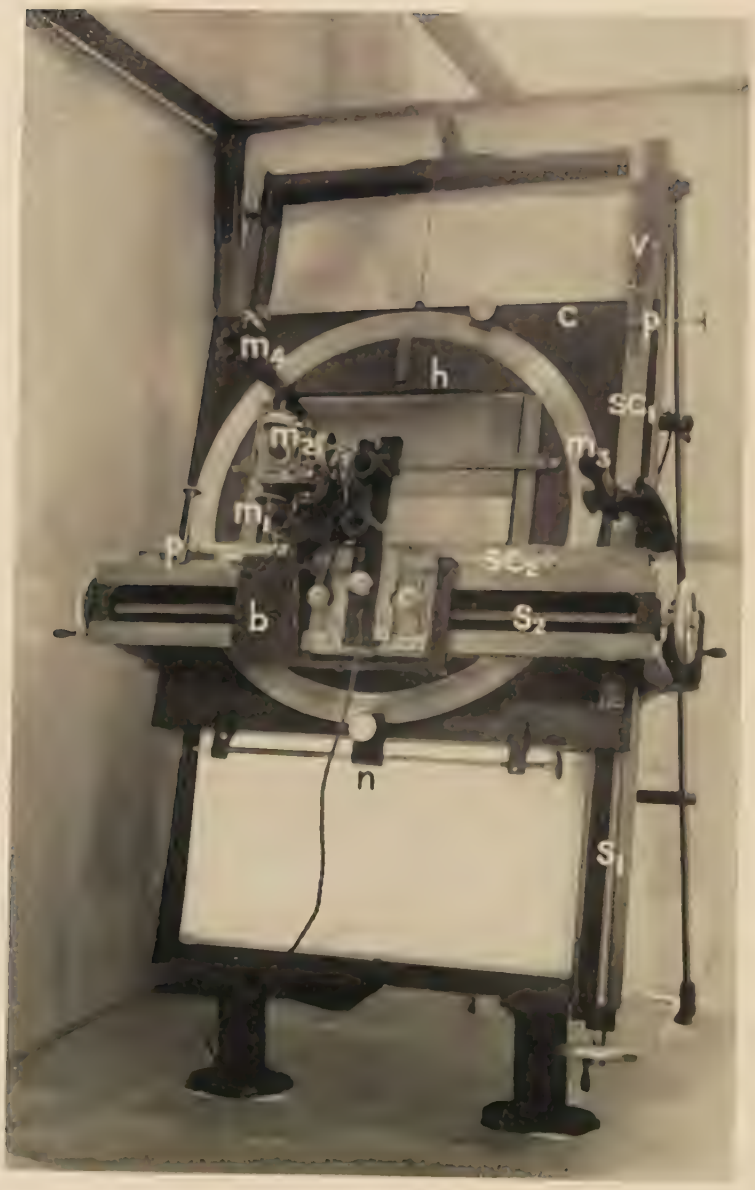

Fig. 1.—Measuring Machine for large Plates.

van Rhijn and Raimond, Measuring Machine for Differential Proper Motions.

INVESTIGATION OF A MEASURING MACHINE AND METHODS FOR MEASURING DIFFERENTIAL PROPER MOTIONS.

P. J. van Rhijn and J. J. Raimond, Jun.

Summary

Section 1. A description of a new measuring machine, now in use at the Kapteyn Laboratory, is given. The machine is of the Repsold type and has accommodation for plates up to a size of 30 × 30 cm.

Section 2. The machine has been tested with respect to three methods of deriving relative proper motions. The principal microscope being alternately adjustable at two different focuses, a pair of plates, both taken in the usual way, can be measured differentially. It appears that measuring through the glass of an ordinary photographic plate does not vitiate the accuracy of the measured distances. The method of the alternating focus is to be preferred to the measuring with respect to the division lines of the scales on account of economy in measurement and reduction.

In *Section* 3 a summary is given of the probable errors of a distance measured on a *Carte du Ciel* plate printed in the Publications of the Kapteyn Astronomical Laboratory.

1. *Description of the Machine*

In 1930 September a new measuring machine was installed at the Kapteyn Astronomical Laboratory, Groningen. The machine is of the Repsold type. It is designed by Mr. G. H. Müller, observer at our laboratory, under supervision of the director, Dr. P. J. van Rhijn, and has been manufactured in the shop of Mr. G. de Koningh at Arnhem.

The apparatus (fig. 1, Plate 12) is inclined at an angle of 15° to the vertical. The plate carriage, c, is guided by an accurately figured V-way, v, on one side and a plane support on the other side of the machine bed. The plate carriage is moved up and down by a screw, s_1, of 0·5 cm pitch, and is firmly pressed against the V-way by means of two powerful springs, provided at their free ends with ball-bearing wheels, running along the back of the V-way. The plate carriage is balanced by means of a massive weight.

The displacement of the carriage is read on a scale, sc_1, by means of a microscope, m_3, attached to the machine bed.

The plate holder, h, is revolvable, and is provided with a position circle, divided into 10′ and read by a microscope, m_4. The plate holder is turned by hand; the accurate orientation is made with a screw, attached to the carriage and operating on a nut, n, which can be clamped to the border of the position circle.

Three plate clamps are so arranged as to accommodate plates up to a

size of 30 × 30 cm. A large adjustable reflecting mirror is supported behind a slit 16 cm wide, cut across the bed. The mirror reflects the daylight into the field of the measuring microscope, m_2.

A heavy horizontal bridge of cast iron is tightly fixed to the machine bed. A block, b, carrying two microscopes moves along the carefully aligned ways of the bridge by means of a screw, s_2, of 0·5 cm pitch. The displacement of b is read by the microscope m_1 on a scale, sc_2, which is secured on the bridge.

The images are pointed with the second microscope m_2, placed vertically above m_1.

Two details deserve special consideration.

1. The scales are movable in a slide by a rack and pinion p, the slide being attached to the machine. This construction allows the adjustment of

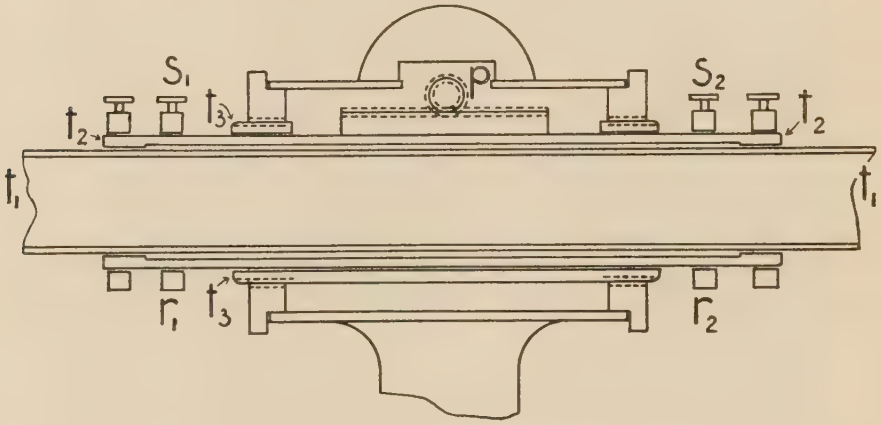

Fig. 2.—*Scheme of the alternating focussing mechanism.*

the scales to the zero-point adopted in the computation of the co-ordinates on the plate. Moreover, the scales being easily removed and replaced by others, we can always use a scale the division of which corresponds with 1′ on the plate.

2. The principal microscope m_2, directed to the star images, can be focussed in the usual way by a rack and pinion. By a simple construction the microscope can be adjusted at two different focal settings.

In fig. 2, t_1 is the tube carrying the micrometer. Tube t_1 is clamped in the tube t_2, which is moved by the rack and pinion p. Two loose rings r_1 and r_2 can be clamped to t_2 with the aid of the screws s_1 and s_2. These rings being once adjusted to the proper position, the microscope can be moved from one focus, defined by r_1, to the other, defined by r_2. The guiding of t_2 in t_3 has been constructed with the utmost care.

2. *The Measurement of Proper Motions*

The methods, usually applied for the determination of relative proper motions, are :

1. The images of both epochs are on separate plates, the new plate being exposed with the glass towards the objective of the telescope.

The plates are put film to film in the measuring engine and the distances between early and new images are measured.

2. The images of both epochs are on separate plates; both plates are exposed in the usual way.

The plates are alternately placed in the machine and the images of both epochs are measured with respect to the same division lines of the scales.

We have tried these two methods with the apparatus described in section 1.

Moreover, a third method, first used by Kreiken,* has been investigated.

3. The plates, both exposed in the usual way, are clamped *glass to film* in the plate holder, *the films of both plates being turned towards the objective of the microscope.* The distance between the early and the corresponding new image is measured in both co-ordinates with a micrometer, the microscope being alternately focussed on the film of the early and the new plate.

In the case of methods 1 and 3 the probable errors of the measured distances, as found by means of the new instrument, are not larger than the values found from former measurements made at our laboratory. These methods can therefore be used advantageously with the present instrument.

Method 2 yields probable errors, which are undoubtedly too large. This is caused by changes of temperature during the observations.

We shall give an account of the measurements made with the new machine to test the methods enumerated above.

Method 1.—The method is tested by remeasuring some ten pairs of images on several days; every day the same distance was measured twice within a short interval of time.

The probable error of measuring the distance between two images is :

$$e = \pm\ 0{\cdot}0010\ \text{mm} = \pm\ 0''{\cdot}06,\dagger \qquad (1)$$

computed from the residuals for each day with respect to the mean of all days, and :

$$e = \pm\ 0{\cdot}0010\ \text{mm} = \pm\ 0''{\cdot}06, \qquad (2)$$

computed from the comparison of the measurements made on the same day.

The latter value is not affected by instabilities of the machine because the measurements are made within a short interval of time.

The means of the residuals of all observations made on the same day are small and can be fully explained by accidental errors; they do not show any pronounced systematic character.

The value (1) being found equal to (2), we conclude that the differences between the observations on different days may be ascribed entirely to errors of pointing.

The probable error found here is smaller than the value $\pm$ 0·0017 mm = $\pm$ 0″·10 usually adopted for the error of measuring a distance between two

* *The Observatory*, **43**, 255, 1920.

† The scale is that of a *Carte du Ciel* instrument : 1 mm = 60″.

images on a *Carte du Ciel* plate. This difference is very probably due to the fact that the observations of some ten selected stars, remeasured several times, are not independent because the observer remembers the shape of the images and the way he has bisected them.*

Method 2.—To test this method with our new machine a plate was put in the plate holder and a number of images were measured with respect to the division lines of the scale sc_2 (fig. 1). The same images were measured on several days, artificially varying the temperature, which was registered by a thermograph.

The measures are satisfactory if the temperature of the room remains constant within about 3° C. In other cases the observations showed systematic changes, which undoubtedly depend on the temperature.

We do not intend to use this method in future observations, because Method 3 is undoubtedly to be preferred.

The microscope m_1 was pointed on a division line of the scale and the star image was bisected by means of the wire of the micrometer m_2. The reading of the micrometer differs from the distance in the X-direction between the image and the division line by a constant quantity which is immaterial. A second measurement was made on each image directly after the first one.

We shall first discuss a series of observations for which the temperature of the room varied between the extreme values 13° and 16° C.

The probable error of the distance between a star image and a division line as computed from the residuals on several days with respect to their mean is equal to the value found from the differences between two consecutive pointings on the same image :

$$r = \pm\ 0{\cdot}0009 \text{ mm} = \pm\ 0''{\cdot}06. \qquad (3)$$

Moreover, the means of the residuals found on the same day do not show a systematic character. We therefore conclude that the measurements are correct if the changes in temperature do not exceed 3° C.

The effect of large variations in temperature may be illustrated by the following figures. During this series of observations the temperature varied over a range of about 15° C.

In this case we find :

$$r = \pm\ 0{\cdot}0010 \text{ mm} = \pm\ 0''{\cdot}06,$$

computed from the differences between two consecutive measurements, and :

$$r = \pm\ 0{\cdot}0060 \text{ mm} = \pm\ 0''{\cdot}36,$$

computed from the residuals on several days with respect to the final mean value.

The mean residuals for each separate day are given in the following Table I :—

* Compare *Gron. Publ.*, 1, 81, 1900, section 13.

TABLE I

Dependence of Mean Residual in Method 2 on the Temperature

Date	Temperature	Mean Residual in mm
1930 Dec. 17	20°	+0·0047
,, ,, 18	6	− 21
,, ,, 18	8	− 16
,, ,, 19	21	+ 41
,, ,, 20	22	+ 56
,, ,, 22	12	− 25
,, ,, 23	8	− 38
1931 Jan. 2	10	− 41

There cannot be any doubt about the existence of an effect depending on the temperature of the room. It is, however, not thought worth while to investigate this point more closely, as Method 3, which is applicable to the same pairs of plates, yields good results and is far more economical.

Method 3.—This method is to be preferred to the former one on account of economy of measurement and reduction. As will be explained below, it can be applied even if the plates are not made of plate glass.

As explained in section 1 the focus of the microscope is changed between the pointing on the early and the corresponding new image.

In order to test the stability of the focussing mechanism the distances between some ten pairs of images were measured several times. Each distance was measured twice each day ; the measurements were repeated on fourteen different days.

The probable error of the distance between two single images computed from the differences between two consecutive observations appears to be the same as the value found from the residuals for each day with respect to the mean of all days, viz. :

$$e = \pm\ 0{\cdot}0008 \text{ mm} = \pm\ 0''{\cdot}05. \qquad (4)$$

This error is slightly better than the value (1) derived according to Method 1 and agrees very well with the error found from the same kind of observations made with our Repsold machine in applying Method 1 (see also Table IV).

We conclude that the alternating focus does not introduce any appreciable error.

The reliability of the method has been tested by means of seven plates chosen at random from our collection of proper motion plates and taken at five different observatories.

As will be explained below, a set of measurements with a known error of pointing is available for these plates ; we call this set R. The same plates have been measured with the new machine using the alternating focussing mechanism ; this set is called K.

The probable error of the measurements K will be computed from the differences between the sets R and K, the probable error of R being known from former measurements. The method of the alternating focus is justified by the fact that the probable error of the method is not larger than might be expected on the basis of the accidental errors of observation.

The measurements R have been performed with respect to the division lines of the scale (Method 2).

Six pairs numbered 1 to 6 in Table II were formerly measured at our laboratory with the Repsold measuring machine, which gives good results.* The plates of the seventh pair being too large for the plate holder of the Repsold machine were measured with our new engine, the temperature being kept constant within 2° C. (see Method 2). The pairs 1, 2, and 7 were measured only in two positions, the remaining ones in four positions of the plate differing 90° respectively.

The set K contains the observations on all seven pairs, using the alternating focus. Pair 1 has been measured twice, the position of the new and early plates being interchanged in the two measurements. With the exception of the pairs 1 and 2 the plates were measured in four positions differing 90°.

For each pair of plates the differences R–K have been represented by the formulæ :

$$\begin{aligned} \Delta x &= a_x x + b_x y + c_x, \\ \Delta y &= a_y x + b_y y + c_y, \end{aligned} \tag{5}$$

where Δx = the distance measured in the x-direction according to Method 2 minus the distance between the same images measured according to Method 3.

Δy = the same quantity in the y-direction.

b_x, a_y, c_x, c_y depend on the difference in relative orientation of one plate with respect to the other for the sets R and K.

a_x and b_y may be due to either or both of the following causes :—

1. A difference in inclination in both sets of measurements of the plate with respect to the plane of the ways guiding the plate holder and the microscope system.
2. The variations of thickness of the plate through which the observations are made, which affect only the set K (see p. 643).

Whatever may be the explanation of these coefficients, the application of linear corrections in computing the probable error is justified by the fact that all linear terms are eliminated from the measures in deriving relative proper motions in the usual way.

The coefficients a_x, b_x, c_x, etc. have in some cases been found by a least square solution ; in other cases they were solved by the method of Christie and Dyson.†

* *G.P.*, 25, 8, 1914 ; *ibid.*, 28, 4, 1918.

† *M.N.*, 55, 61, 1895.

TABLE II

Probable Error of Pointing according to Method 3

Pair No.	Plate Nos.	Centre of Plates		Object	Observatory	Number of Images per Epoch	Number of Stars Measured	Probable Error of the Distance between Two Images				Original Observations discussed in:
		α 1900	δ 1900					R–K	R	K Comp.	K Obs.	
		h m	° ′					mm	mm	mm	mm	
1	F3 / C78	8 35	+20 1	Praesepe	Poulkowa	1	243	± 0·0023	[± 0·0017]	± 0·0016	...	G.P. 41
1						1	243	22	[17]	15	...	G.P. 41
2	1972 / 9266	8 35	+20 1	Praesepe	Greenwich	1	234	24	[17]	18	...	G.P. 41
3	1205 / 9319	10 55	−44 58	...	Cape	2	125	29	20	20	± 0·0018	G.P. 28
4	1313 / 9321	11 55	−44 59	...	Cape	2	119	30	21	22	21	G.P. 28
5	1531 / 2155	4 16	+14 0	Hyades	Bordeaux	3	152	39	27	28	23	Unpubl.
6	1527 / 2156	4 24	+14 0	Hyades	Bordeaux	3	135	35	27	22	24	Unpubl.
7	1379 / 32262	8 35	+20 1	Praesepe	Allegheny	1	99	46	[30]	35	31	Unpubl.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	

Pairs 1 to 6 are *Carte du Ciel* plates ; size 16 × 16 cm, scale 1 mm = 60″.
Pair 7 : size 20 × 25 cm, scale 1 mm = 14″·65.
On the *Carte du Ciel* plates images with a distance from the centre exceeding 60′ have been excluded from the reduction.

We next compute the probable values of the residuals:

$$\begin{aligned} &\Delta x - (a_x x + b_x y + c_x), \\ &\Delta y - (a_y x + b_y y + c_y). \end{aligned} \tag{6}$$

These probable values are reduced to the values they would have had if each distance had been derived from the measurement of a single pair of images in one position of the plate. The result is inserted in column (9) of Table II. The mean of the values in the x- and y-direction are given.

The probable error of a distance between a single pair of images of the set R is given in the tenth column of the table. It has been derived for the pairs 3 to 6 from the differences between the distances measured in two positions differing 180°. In the other cases no measurements differing by 180° are available. The amount ± 0·0017 mm adopted for the pairs 1 and 2 is taken in accordance with former measurements at this laboratory (see Table IV). The value ± 0·0030 mm for pair 7 is computed from the data given by Schlesinger.*

The probable errors of the set K given in the eleventh column of Table II are easily found from the preceding columns R–K and R.

In the case of the pairs 1 and 2 the probable errors of the observations K appear on the average to amount to ± 0·0017 mm, which is equal to the value found in former investigations at this laboratory (see Table IV).

In the case of pairs 3 to 6 the probable error of the set K is a little larger: $\epsilon_0 = \pm$ 0·0023 mm on the average. This, however, is not due to errors introduced by the new method of measurement. For the probable errors derived from the comparison of the observations in two positions of the plate differing 180° are a little larger as well:

$$\epsilon_0 = \pm \text{ 0·0024 mm for the set R (column 10).}$$

Moreover, a part of the difference between the value of ϵ_0 for the pairs 1 and 2 and the pairs 3 to 6 is due to the fact that the pairs 3 to 6 contain two or three images per epoch and the pairs 1 and 2 only one image. The probable errors derived from the mean of two or three images have been multiplied by $\sqrt{2}$ and $\sqrt{3}$ respectively in order to make them comparable with the probable error of the pairs 1 and 2. It will be shown in the next section that for the measures of our plates these factors are probably too large.

In the case of the Allegheny pair 7 the probable error of the set K in the thirteenth column is somewhat larger than the probable error computed from Schlesinger's data and than the value derived from the difference between two positions of the plate, differing by 180°, given in the twelfth column. Quadratic terms † appear to be present in the residuals $\Delta y - (a_y x + b_y y + c_y)$. After applying corrections for these terms, the probable error K of the

* *Allegheny Publications*, 4, 13, 1919.

† The quadratic terms are probably due to the curvature of the plates.

Method 2 is reduced to a value smaller than the normal value ± 0·0030 mm. See Table III, where the "uncorrected" errors are given in parentheses.

TABLE III

The Probable Error of Pointing of Pair 7

	R–K	R	K
x-direction	± 0·0036 mm	± 0·0030 mm	± 0·0020 mm
y-direction	44(55) ,,	30 ,,	32(46) ,,
Mean	40(46) ,,	30 ,,	23(35) ,,

The probable errors K obs. derived from the intercomparison of two positions of the plate differing 180° are in the twelfth column of Table II (Method 3). They agree with the corresponding figures in the column R (Method 2).

The method of the alternating focus has further been tested by dividing each plate into 9 rectangular areas and computing the mean value of the residuals (6) for each area. Although in some cases there was a slight indication of quadratic terms, the mean residuals were always small and could be explained as accidental errors.

The conclusion is that for all seven pairs the probable errors of the observations K are not larger than might be expected as a consequence of the unavoidable errors of observation ; the method of measuring relative proper motions by means of the alternating focus mechanism is therefore unobjectionable.

It is to be remembered that the plates are not manufactured of plate-glass. We therefore must expect variations as large as 0·3 mm in the thickness of the glass and hence a change in the focus which might increase the errors of observation. It appears from the present investigation that these errors are negligible. Probably the thickness varies approximately linearly with the co-ordinates on the plate, and will therefore introduce linear terms in the measured distances which are eliminated in the reduction.*

The local variations of thickness are not larger than about 0·05 mm. The effect of these variations has an accidental character ; the maximum value of the deviations may be estimated at 0″·01 in the measured distance, which is entirely negligible.

The method of the alternating focussing mechanism is to be preferred to the method of measuring the position of the images with respect to the division lines of the instrument on account of economy of measurements and reductions, the time required for the entire derivation of the proper motions being reduced in the ratio 2 : 5.

* See also p. 640.

3. *Probable Errors of the Distance between Two Images Measured on a* Carte du Ciel *Plate.*

Table IV gives a summary of the probable errors found at the Kapteyn Astronomical Laboratory on plates taken with a *Carte du Ciel* refractor. Unless otherwise stated, the probable errors are reduced to the value for the distance of a single pair of images measured in one position of the plate. The figures given with an asterisk are not printed in the volume quoted in the second column of Table IV ; they are computed from the original data.

TABLE IV

The Probable Errors of a Distance between Two Images measured on a Carte du Ciel *Plate at the Kapteyn Laboratory*

Method	Gron. Publ. No. and Page	Observatory	No. of Images per Epoch	P.E. of the distance between a Single Pair of Images			Object
				e	ϵ_0	ϵ	
				mm	mm	mm	
a. Latent images	1 ; 83, 76, 86	Helsingfors	3	...	±0·0016 (3)	±0·0025	Parallaxes
	10 ; 34	,,	3	...	...	24	,,
	14 ; 18	,,	3	...	...	25	Pr. mot. Hyade
	19 ; 31, 32	,,	3	...	±0·0017 (3)	26	Pr. mot.
	20 ; 24, 27	,,	3	...	22 (3)	19	Parallaxes
				...	±0·0018	±0·0024	
. New plate taken through the glass	26 ; 3, 17, 18	Potsdam	1	±0·0012 (1)	±0·0017 (1)	±0·0022	Pr. mot. Praese
	33 ;	,,	1	...	17*(1)	...	Pr. mot.
	39 ; 11	Helsingfors	3	...	17 (1)	± 24	Pr. mot.
	Unpublished	Alger	2	± ·0008 (1)	17 (1)	...	Pr. mot. S.A., 6
	This paper, p. 637.			10 (1)	...	...	
				±0·0010	±0·0017	±0·0023	
. Measuring with respect to the division lines	25 ; 9, 17	Helsingfors	1	±0·0016 (1)	±0·0017*(1)	±0·0025	Pr. mot.
			3	18 (3)	21*(3)		
	28 ; 4	Cape	2	18 (2)	20*(2)	...	Pr. mot.
	30 ; 23	Helsingfors	3	...	20*(3)	±0·0027	Pr. mot.
	35 ; 1, 4	,,	3	± ·0016 (3)	...	27	Pr. mot. Hyade
	Unpublished	Bordeaux	3	19 (3)	26 (3)	28	Pr. mot. Hyade
	This paper, p. 638.			13 (1)	...	...	
				±0·0017	±0·0021	±0·0027	
. Measuring with the alternating focus	This paper, p. 641.	Cape	2	±0·0010 (1)	±0·0019 (2)	...	
	This paper, p. 641.	Bordeaux	3	11 (1)	23 (3)	...	
	Kreiken †		3	...	20 (3)	± ·0024	
	This paper, p. 639.			± ·0008 (1)	...	...	
				±0·0010	±0·0021	±0·0024	

† *The Observatory, loc. cit.*

Three kinds of errors are tabulated :

e : the pure error of pointing derived from a large number of measurements of a small number of distances.

In the case of Method 2, $e = r\sqrt{2}$, the meaning of r being defined on p. 638.

ϵ_0 : the pure error of pointing derived from the differences between the observations made in two positions of the plate differing by 180° or from the differences between two series of measurements in the same position of the plate.

ϵ : the total probable error of the distance between a single pair of images measured in two positions of the plates found by a comparison with meridian proper motions or from the differences of two or more plates covering the same region of the sky. ϵ contains the error of pointing, the "plate error" and the errors of the reduction.

The probable errors, expressed in mm, are grouped according to the method of observation.

The data relating to Method 3 are necessarily scarce, as no larger programmes have been observed with this method.

We now derive a final mean value for each of these three probable errors e, ϵ_0, and ϵ.

1. *The Value of e.*—The figures between parantheses in the fifth column of Table IV indicate that the probable error has been computed from a single residual (1) or from the mean of two (2) and three (3) residuals respectively. In both latter cases the values found from the original data have been multiplied by $\sqrt{2}$ and $\sqrt{3}$ respectively in order to reduce them to the value for a single pair of images. It appears from Table IV that the values of e derived from two or three pairs of images are always larger than the value found from a single pair of images ; the factors $\sqrt{2}$ and $\sqrt{3}$ are therefore a little too large.

We find for Method 2 :

$e = \pm$ 0·0014 mm derived from a single pair of images,
$e = \pm$ 0·0018 mm derived from the mean of two or three pairs of images.

The difference is corroborated if we compute the value of e first from the separate pairs of images and secondly from the mean of two or three pairs of images in the same series of measurements :

$e = \pm$ 0·0013 mm from separate pairs,
$e = \pm$ 0·0017 mm from the mean of two or three pairs.

The value of e found from the mean of two or three pairs of images have therefore been diminished by 20 per cent. We then find the mean values :

$e = \pm$ 0·0014 mm for Method 2,

$e = \pm$ 0·0010 mm for Methods 1*a*, 1, and 3.

The error e only serves as a criterion for the stability of the machine during the measurements and cannot be used as a measure of the accuracy of the final results.

2. *The Value of* ϵ_0.—Again the values found from the mean of two or three pairs of images are larger than the values found from a single pair of images :

$\epsilon_0 = \pm$ 0·0017 mm derived from a single pair of images,
$\epsilon_0 = \pm$ 0·0021 mm derived from the mean of two or three pairs of images.*

The same fact is shown in Table V. The second column gives the probable error derived from the mean of two or three pairs of images, the third column the value derived from each separate pair on the same plate.

The probable error of pointing derived from the mean of two or three images therefore seems to be 20 per cent. larger than the value found from separate pairs of images.

TABLE V

Probable Error of Pointing ϵ_0 *computed from the Mean of Two or Three Images and from a Single Pair of Images*

Number of Images per Epoch	ϵ_0		Method of Measurement
	Computed from the Mean of 2 or 3 Pairs	Computed from Separate Pairs	
2	± 0·0020 mm	± 0·0017 mm	Alternating focus.
2	22 ,,	17 ,,	
3	25 ,,	21 ,,	
3	28 ,,	23 ,,	Division lines.
3	19 ,,	17 ,,	Through the glass.
Mean	± 0·0023 mm	± 0·0019 mm	

Applying this correction to the probable errors derived from two or three pairs of images we find from Table IV that the probable error of pointing of the distance between a single pair of images on a *Carte du Ciel* plate is :

$$\epsilon_0 = \pm\ 0{\cdot}0017 \text{ mm} = \pm\ 0''{\cdot}10.$$

3. *The Value of* ϵ.—The mean value found in Table IV for Method 2 is somewhat larger than the values for the other methods. But the difference is small and we may adopt the mean value :

$$\epsilon = \pm\ 0{\cdot}0025 \text{ mm} = \pm\ 0''{\cdot}15$$

* We have excluded the value ϵ_0 of *Groningen Publications*, 1, because the distances in this publication have been derived from a large number of pointings.

for the total probable error of a distance between a single pair of images measured in two positions of a *Carte du Ciel* plate. This value holds for images at a distance from the centre of the plate not exceeding 60′.

The value of ϵ given for the Method 3 depending on a single plate is not conclusive. The results obtained in section 2, however, have established the equivalence of Methods 2 and 3. It follows that the probable error $\pm$ 0·0025 mm holds as well for the Method 3 as for the other methods.

Our thanks are due to Mr. W. Ebels, observer at the Kapteyn Astronomical Laboratory, who made the measurements and reductions.

THE PROPER MOTIONS OF STARS IN THE REGION OF CLUSTER M37 (N.G.C. 2099).

Violet F. White, M.Sc.

Two plates centred on the galactic cluster in Auriga, M37 (5^h 46^m, 32° 31′ (1900)), were taken with the Wilson reflector at an interval of 35 years, one obtained in 1895 by Mr. W. E. Wilson with an exposure of 40 minutes, and the other in 1930 by Mr. C. C. L. Gregory with an exposure of 60 minutes. The plates were of very good quality and showed measurable circular images for stars whose magnitudes ranged from 9 to 13 on the Harvard Visual Scale, which is the magnitude scale used throughout this paper.

The cluster is classified by Melotte * in Class II, which comprises "Loose clusters having a regular well-defined outline," and is further described by him as "A fine example of Class II. Condenses well towards centre."

The diameter of the cluster was taken to be 20′, within which limit the stars showed a marked condensation; the whole region measured had a diameter of 50′ and included 223 stars—126 in the cluster and 97 outside it. The 8 brightest stars, magnitude 10, within the cluster area were not considered to belong to the cluster.

Both plates were photographed directly, instead of one through the glass as in the other Wilson regions. The plates were measured separately in conjunction with the same réseau, the stars on either plate having approximately the same positions relative to the réseau squares, and the measurements were combined so that the same differential method of reduction could be used as in Regions I, II, III.† Plate constants were calculated by the method of Christie and Dyson from 32 of the faintest stars, mean magnitude 12·5, which were evenly distributed on the plate and mostly

* *Mem. R.A.S.*, **60**, 175, 1915.
† *M.N.*, **92**, 519, 1932; **93**, 105, 1932.

outside the limits of the cluster. Proper motions, μ_x, μ_y, relative to these comparison stars were determined from the equations

$$\mu_x = M_x + a + bx + cy, \qquad \mu_y = M_y + d + ex + fy. \tag{1}$$

Two complete reductions, (*a*) and (*b*), were made, each based on the mean of 3 measurements, and the values of the constants obtained are given in Table I.

TABLE I

	(*a*)	(*b*)
a	−0·0042	−0·0033
b	0·0003	0·0004
c	0·0016	0·0022
d	0·0012	−0·0016
e	−0·0008	−0·0013
f	0·0010	0·0009

The two sets of constants are not identical, as the plates were necessarily moved slightly relative to the réseau between the two reductions. The unit

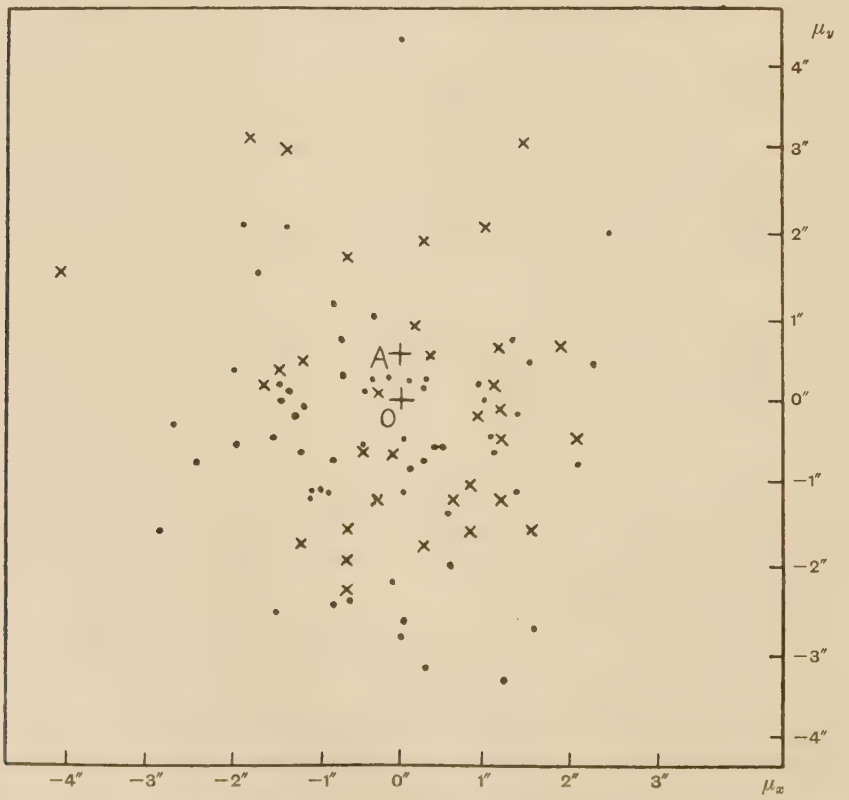

FIG. 1.—*Centennial proper motions of 96 stars outside the cluster. Dots refer to stars with mean mag. 12. Crosses refer to stars with mean mag. 10.*

used in equations (1) for x, y, M_x, M_y was 0·5 cm., and it was possible to measure one ten-thousandth of this unit, so that μ_x, μ_y were obtained as

multiples of the scale unit 0·00005 cm. :

1 scale unit in x in 35 years $\equiv 0''\cdot11$ per century in α.
1 ,, ,, y ,, ,, $\equiv 0''\cdot09$,, ,, ,, δ.

The diameters of all the stars were measured and the corresponding magnitudes 10 to 13 obtained from von Zeipel and Lindgren's catalogue,* which gives visual magnitudes of 2110 stars in the region, determined by direct comparison with the Polar Sequence. The 10 B.D. stars in the region of about magnitude 9 were reduced to the Harvard Visual Scale by applying the correction given by A. Pannekoek.†

The proper motions discussed are the mean values of the two determinations (*a*) and (*b*), and the probable errors of these mean values calculated

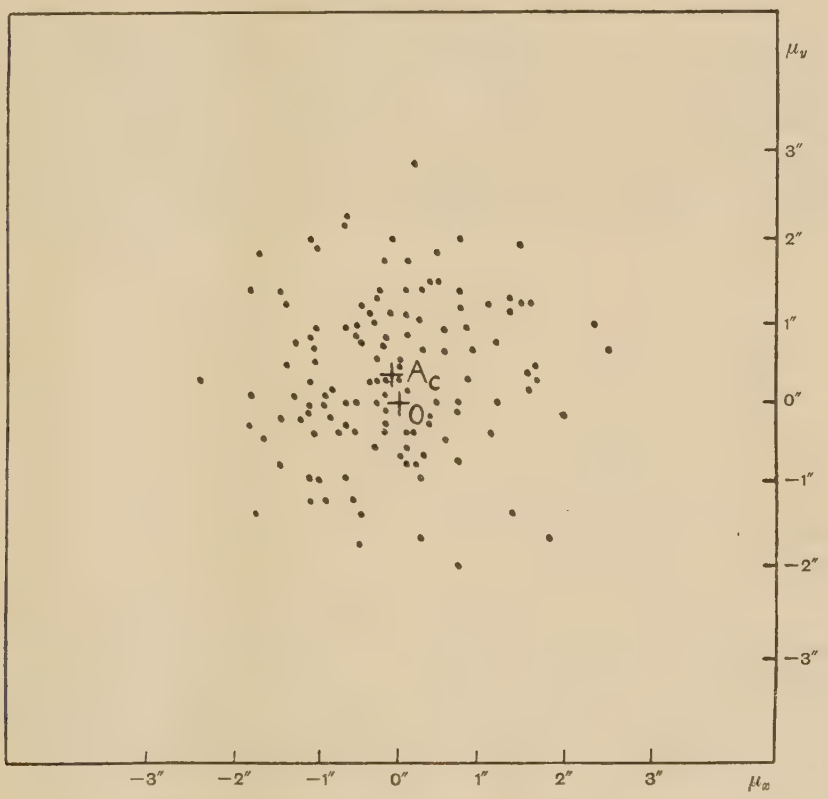

FIG. 2.—*Centennial proper motions of 126 stars in the cluster. Mean mag. 12·2.*

from the deviations of the separate proper motions (*a*) and (*b*) from their means were $\pm$ 0″·3 per century.

The proper motions, represented by vectors drawn from O with a dot marking the other end of the vector, are plotted in figs. 1 and 2. The proper motions for stars outside the cluster and inside the cluster are drawn in separate diagrams, but they refer to the same origin, O, and are on the same

* *Kungl. Sven. Vetens. Akad. Handl.*, **61**, No. 15, 1921.
† *Pubn. Astr. Inst. Univ. Amsterdam*, **1**, Table 26, 1924.

scale. Fig. 2 shows that the proper motions in the cluster are small and evenly distributed about their mean value, and a comparison of figs. 1 and 2 shows that the cluster as a whole has a small proper motion relative to the stars outside the cluster.

One B.D. star with large proper motion was outside fig. 1 ; its centennial proper motions determined from meridian observations (Schorr, *Eigenbewegungs-Lexikon*) and from the measurements relative to the comparison stars were :

		Catalogue		Measured	
B.D.	Mag.	μ_α	μ_δ	μ_α	μ_δ
32°·1102	8·8	$0^s{\cdot}03$	$-13''{\cdot}7$	$0^s{\cdot}36$	$-12''{\cdot}7$

The proper motions relative to the comparison stars, of the stars outside the cluster, were converted into absolute proper motions by comparing the mean relative proper motions and mean parallaxes of groups of stars, as described in my previous papers. The analysis, in terms of seconds of arc per century, is given in Table II. The mean proper motion of the cluster relative to the comparison stars is also added, and the proper motion of the stars outside the cluster relative to the cluster is well shown in column $\bar{\mu}_y$.

Table II

Region	Mean Mag.	No. of stars	H_m	P_x	$\bar{\mu}_x$	$P_x-\bar{\mu}_x$	P_y	$\bar{\mu}_y$	$P_y-\bar{\mu}_y$
			"	"	"	"	"	"	"
IV_{12}	12·0	61	0·7	0·0	−0·4	0·4	−0·6	−0·5	−0·1
IV_{10}	10·0	36	1·3	0·0	0·1	−0·1	−1·2	−0·4	−0·8
IV	11·2	97	0·9	0·0	−0·1	0·1	−0·8	−0·4	−0·4
Comparison stars	12·5	32	0·6	0·0	0·0	0·0	−0·5	0·0	−0·5
Cluster stars	12·2	126	...	...	−0·1	...	...	0·4	...

The correction 0″·0, −0″·5, deduced from the comparison stars, was applied to the relative proper motions to convert them to absolute proper motions ; *i.e.* in fig. 1 the origin, A, of absolute proper motions has co-ordinates 0″·0, 0″·5 referred to O, the origin of relative proper motions.

The absolute centennial proper motion of the cluster determined from this analysis is :

$$\mu_\alpha = -0''{\cdot}1, \qquad \mu_\delta = -0''{\cdot}1.$$

The parallax of the cluster calculated by von Zeipel and Lindgren * from the apparent magnitudes of stars of different spectral types is 0″·0007, which is the value adopted by Shapley ; † the value determined by Raab ‡ is 0″·0005. From these values it was to be expected that the proper motion of the cluster would be very small.

* *Kungl. Sven. Vetens. Akad. Handl.*, **61**, No. 15, 1921.
† *Harv. Obs. Monogr.*, No. 2, 1930.
‡ *Lund. Medd.*, ser. ii, **3**, 1922.

If the absolute proper motion of the cluster is considered to be zero, then the correction necessary to convert the relative proper motions into absolute proper motions is $0''{\cdot}1$, $-0''{\cdot}4$, in close agreement with the value $0''{\cdot}0$, $-0''{\cdot}5$ given by the parallax analysis. This value $0''{\cdot}1$, $-0''{\cdot}4$ agrees exactly with the value given by the parallax analysis of all the stars outside the cluster, and gives an origin of absolute proper motions, A_c, with co-ordinates $-0''{\cdot}1$, $0''{\cdot}4$ referred to O.

The number of stars, r, with absolute proper motions in the directions

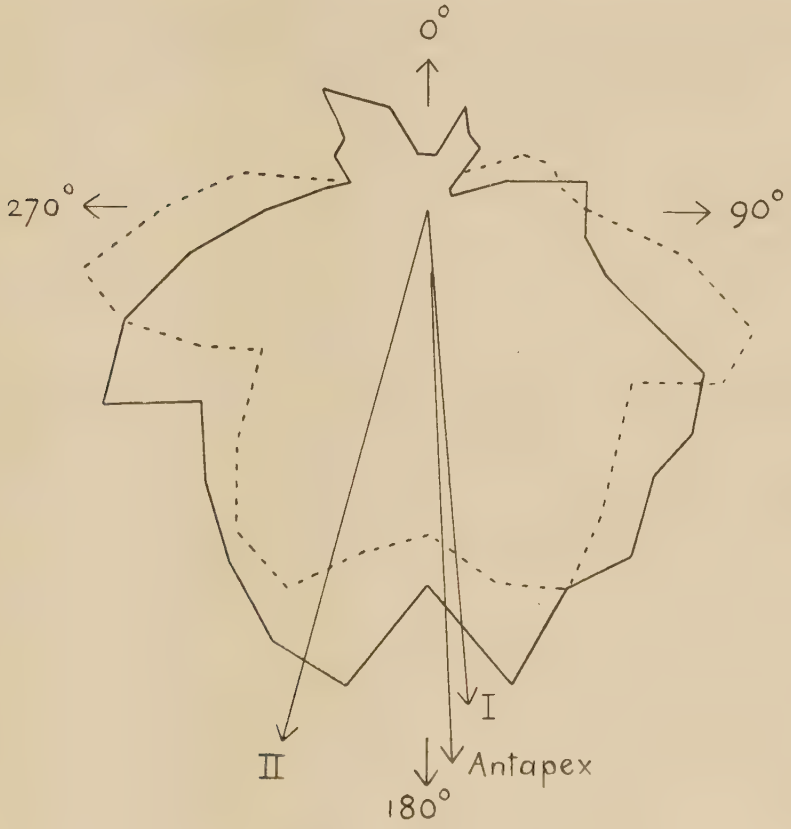

Fig. 3.—*Region IV. 97 stars. Mean mag.* 11·2.

of various position angles, θ, are given in Table III; the numbers for Regions IV, IV_{12} and IV_{10} are calculated from origin A, and for the cluster from origin A_c.

The drift curve is drawn from the values for Region IV in fig. 3. The directions of Drifts I and II and the solar antapex, calculated from their positions assumed in *M.N.*, **92,** 522, 1932, are all near the direction 180°, having the values 175°, 196° and 177° respectively.

The dotted curve in fig. 3 shows the small alteration in the curve which results from calculating the absolute proper motions from the origin A_c, instead of from A.

TABLE III.

θ	IV	IV_{12}	IV_{10}	Cluster
	r			
°				
0	1	1	0	3
10	1	0	1	4
20	0	0	0	4
30	3	0	3	4
40	0	0	0	4
50	0	0	0	3
60	1	1	0	6
70	0	0	0	2
80	2	1	1	2
90	4	2	2	5
100	0	0	0	2
110	2	1	1	2
120	5	3	2	4
130	5	3	2	1
140	3	2	1	5
150	5	3	2	0
160	7	3	4	6
170	3	2	1	3
180	8	7	1	1
190	3	1	2	6
200	7	2	5	2
210	7	6	1	4
220	1	1	0	5
230	5	5	0	4
240	5	5	0	4
250	4	4	0	7
260	3	1	2	5
270	2	1	1	3
280	1	0	1	1
290	1	1	0	3
300	1	1	0	3
310	2	2	9	2
320	1	1	0	4
330	3	1	2	3
340	1	0	1	4
350	0	0	0	5

The distribution of the proper motions in the cluster is almost circular, and if the internal proper motions are assumed to represent errors of measurement only, then the probable error of the proper motions is $\pm 0''{\cdot}7$ per century in both μ_x and μ_y.

The number of stars in the region with absolute centennial proper motions, μ, between various limits are given in Table IV.

TABLE IV

μ	IV	IV_{12}	IV_{10}	Cluster
	No. of stars			
" "				
0·0 – < 1·0	13	9	4	41
1·0 – < 2·0	42	25	17	71
2·0 – < 3·0	28	17	11	14
3·0 – < 4·0	11	9	2	0
4·0 – < 5·0	2	1	1	0
5·0 and greater	1	0	1	0

I thank Professor H. Hilton, Mr. C. C. L. Gregory and Dr. W. M. Smart for their help throughout this work.

Bedford College,
London, N.W. 1.

OBSERVATIONS OF NOVÆ, 1932–1933.

W. H. Steavenson.

The following estimates of magnitude were made with the 20-inch Hindle reflector.

References to the comparison stars used were given in *M.N.*, **92**, 720.

From 1932 August 13 to October 12, inclusive, there is a gap in the observations, due partly to the meeting of the I.A.U. at Cambridge, Mass., and partly to absence of the tube of the telescope during its remodelling by Mr. Hindle.

Nova Aquilæ (1918)

Observed on one night only, 1932 May 30, when the magnitude was estimated to be 10·8.

Nova Aurigæ (1892)

Observed on three nights.

Date	Mag.
1932 Oct. 28	14·9
Nov. 5	14·8
26	14·8

Nova Cygni (1876)

Observed on twenty-four nights. The usual amount of slight variability was noted, the total range shown by the observations being $0^m.76$.

Date	Mag.	Date	Mag.	Date	Mag.
1932 May 30	15·00	1932 July 13	15·00	1932 Aug. 8	14·76
June 9	14·61	25	14·71	10	14·71
21	14·96	30	14·45	12	14·76
July 1	15·00	31	14·44	Oct. 17	15·10
4	15·10	Aug. 4	14·58	Nov. 26	15·00
7	15·10	5	14·60	28	15·00
9	15·10	6	14·65	Dec. 3	15·00
10	15·20	7	14·76	20	15·10

Nova Cygni (1920)

Observed on one night, 1932 May 30, when the magnitude was estimated to be 15·2. There appears to have been little, if any, fading since the close of the previous series of observations in 1931 October, and the nova is probably approaching a stationary condition, if this has not already been reached.

Nova Lacertæ (1910)

Observed on two nights.

Date	Mag.
1932 June 14	14·4
Nov. 5	14·3

It is perhaps doubtful whether this nova has exhibited any certain variability for a considerable number of years.

Nova Ophiuchi (1848)

Observed on thirteen nights. The changes recorded are unusually small for this object, the range being only $0^{m}.2$. Generally it approximates to a whole magnitude. When best seen the nova is not quite sharply defined, though it seems to have a stellar nucleus.

Date	Mag.	Date	Mag.
1932 May 30	12·4	1932 July 1	12·5
June 9	12·4	8	12·5
12	12·3	9	12·5
14	12·3	10	12·5
16	12·5	30	12·5
21	12·3	Aug. 5	12·5
22	12·3		

Nova Persei (1901)

Observed on sixty-five nights. The usual marked variability has been maintained, the range of brightness being $1^{m}.51$. The observations in this series suggest a period of about eight months, the mean brightness in 1932 November and December being considerably less than in 1932 July and 1933 March; but nothing periodic can be made out of the more rapid oscillations. The wisp of nebulosity attached to the nova on its south preceding side still persists, and was well seen on several occasions.

Date	Mag.	Date	Mag.	Date	Mag.
1932 July 7	12·53	1932 Dec. 6	13·88	1933 Feb. 15	13·00
13	12·37	7	13·77	23	13·10
30	12·60	14	13·62	28	12·90
Aug. 10	12·95	15	13·67	Mar. 12	12·79
12	12·79	17	13·88	14	12·90
Oct. 13	13·62	18	13·52	21	12·87
17	13·57	19	13·57	22	13·05
18	13·57	20	13·42	23	12·95
28	13·57	22	13·36	24	13·00
31	13·73	24	13·57	25	12·95
Nov. 3	13·82	30	13·85	26	12·95
5	13·82	31	13·47	27	12·95
21	13·52	1933 Jan. 1	13·67	28	12·90
22	13·57	3	13·88	30	13·26
23	13·57	4	13·72	Apr. 1	13·30
24	13·57	5	13·67	7	12·95
26	13·83	6	13·02	13	13·52
28	13·77	21	13·82	14	13·57
29	13·88	25	13·77	15	13·42
Dec. 1	13·78	Feb. 2	13·82	17	13·36
3	13·83	10	12·92	18	13·47
4	13·67	14	12·90		

PHOTOMETRIC OBSERVATIONS OF NOVA PERSEI, 1901.

B. M. Peek, M.A.

The following visual estimates of the magnitude of Nova Persei, 1901, are a continuation of the series last published in *M.N.*, **91**, 203, 1930, the period covered being from 1930 September 15 to 1933 March 6.

Commencing J.D. 2426650, revised magnitudes have been adopted for Seares' comparison stars numbers 8 and 16, as agreed in consultation with Dr. W. H. Steavenson. For 8, $13^{m}.25$ replaces $13^{m}.16$, and for 16, $13^{m}.10$ replaces $13^{m}.20$. Since before this revision the brightness of the two was generally averaged, no discontinuity in the estimated magnitudes is likely to appear.

J.D. [2426]	Mag.	J.D. [2426]	Mag.	J.D. [2426]	Mag.
235·36 (1930)	13·08	622·37 (1931)	13·40	979·39 (1932)	12·87
238·32	13·18	623·34	13·32	982·41	13·10
244·34	13·25	625·37	13·67	983·36	13·08
245·36	13·18	626·49	13·62	984·35	12·93
250·35	13·23	628·45	13·58	989·37	13·10
256·35	13·02	629·41	13·58	994·45	13·67 ±
259·30	13·05 ±	630·48	13·60	[2427]	
262·30	12·75	631·32	13·62	003·35	13·59
266·40	12·64	635·49	13·37	004·40	13·57
268·49	12·06	636·36	13·47 ±	007·36	13·91
269·36	12·32	645·30	13·37 ±	009·49	13·74
275·35	13·47	647·26	13·28	019·37	< 13·7
291·27	13·28	648·30	13·28	035·25	13·67
296·36	13·35	650·29	13·00	038·32	13·87
305·25	13·37	651·27	13·00	040·27	13·62
306·26	13·42	653·26	12·98	043·35	13·84
319·24	13·52	655·29	13·00	045·30	13·82
331·26	13·60	657·39	12·87	049·27	13·67 ±
337·24	13·77	658·39	12·97	060·25	13·59
338·26	13·67	665·27	12·77	069·31	13·38
351·40 (1931)	13·72	684·30	13·54	076·25 (1933)	13·52
355·24	13·77	718·37 (1932)	12·95	079·25	13·38
366·28	13·97	736·25	13·57	094·46	13·57
383·31	13·31	745·30	13·05	098·44	13·54
409·35	13·02	...	...	099·46	13·54
417·41	13·02	970·40	13·10 ±	100·40	13·54
427·38	12·85 ±	972·37	13·10	117·35	12·78
454·42	< 13·2	974·46	12·95	123·35	13·08
...	...	976·35	12·95	131·30	12·91
619·35	13·37	978·46	12·92	138·31	13·00

Erratum

P. 320, in list of presents received, *for* "Greenwich Apparent Noon" *read* "Greenwich Mean Noon."